Graduate Texts in Mathematics

Sheldon Axler
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Harmonic Function Theory

Second Edition

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Harmonic Function Theory

Second Edition

With 21 Illustrations

Springer

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Contents

Preface

Harmonic functions—the solutions of Laplace's equation—play a crucial role in many areas of mathematics, physics, and engineering. But learning about them is not always easy. At times the authors have agreed with Lord Kelvin and Peter Tait, who wrote ([18], Preface)

> There can be but one opinion as to the beauty and utility of this analysis of Laplace; but the manner in which it has been hitherto presented has seemed repulsive to the ablest mathematicians, and difficult to ordinary mathematical students.

The quotation has been included mostly for the sake of amusement, but it does convey a sense of the difficulties the uninitiated sometimes encounter.

The main purpose of our text, then, is to make learning about harmonic functions easier. We start at the beginning of the subject, assuming only that our readers have a good foundation in real and complex analysis along with a knowledge of some basic results from functional analysis. The first fifteen chapters of [15], for example, provide sufficient preparation.

In several cases we simplify standard proofs. For example, we replace the usual tedious calculations showing that the Kelvin transform of a harmonic function is harmonic with some straightforward observations that we believe are more revealing. Another example is our proof of Bôcher's Theorem, which is more elementary than the classical proofs.

We also present material not usually covered in standard treatments of harmonic functions (such as [9], [11], and [19]). The section on the Schwarz Lemma and the chapter on Bergman spaces are examples. For

completeness, we include some topics in analysis that frequently slip through the cracks in a beginning graduate student's curriculum, such as real-analytic functions.

We rarely attempt to trace the history of the ideas presented in this book. Thus the absence of a reference does not imply originality on our part.

For this second edition we have made several major changes. The key improvement is a new and considerably simplified treatment of spherical harmonics (Chapter 5). The book now includes a formula for the Laplacian of the Kelvin transform (Proposition 4.6). Another addition is the proof that the Dirichlet problem for the half-space with continuous boundary data is solvable (Theorem 7.11), with no growth conditions required for the boundary function. Yet another significant change is the inclusion of generalized versions of Liouville's and Bôcher's Theorems (Theorems 9.10 and 9.11), which are shown to be equivalent. We have also added many exercises and made numerous small improvements.

In addition to writing the text, the authors have developed a software package to manipulate many of the expressions that arise in harmonic function theory. Our software package, which uses many results from this book, can perform symbolic calculations that would take a prohibitive amount of time if done without a computer. For example, the Poisson integral of any polynomial can be computed exactly. Appendix B explains how readers can obtain our software package free of charge.

The roots of this book lie in a graduate course at Michigan State University taught by one of the authors and attended by the other authors along with a number of graduate students. The topic of harmonic functions was presented with the intention of moving on to different material after introducing the basic concepts. We did not move on to different material. Instead, we began to ask natural questions about harmonic functions. Lively and illuminating discussions ensued. A freewheeling approach to the course developed; answers to questions someone had raised in class or in the hallway were worked out and then presented in class (or in the hallway). Discovering mathematics in this way was a thoroughly enjoyable experience. We will consider this book a success if some of that enjoyment shines through in these pages.

Acknowledgments

Our book has been improved by our students and by readers of the first edition. We take this opportunity to thank them for catching errors and making useful suggestions.

Among the many mathematicians who have influenced our outlook on harmonic function theory, we give special thanks to Dan Luecking for helping us to better understand Bergman spaces, to Patrick Ahern who suggested the idea for the proof of Theorem 7.11, and to Elias Stein and Guido Weiss for their book [16], which contributed greatly to our knowledge of spherical harmonics.

We are grateful to Carrie Heeter for using her expertise to make old photographs look good.

At our publisher Springer we thank the mathematics editors Thomas von Foerster (first edition) and Ina Lindemann (second edition) for their support and encouragement, as well as Fred Bartlett for his valuable assistance with electronic production.

CHAPTER 1

Basic Properties of Harmonic Functions

Definitions and Examples

Harmonic functions, for us, live on open subsets of real Euclidean spaces. Throughout this book, n will denote a fixed positive integer greater than 1 and Ω will denote an open, nonempty subset of $\mathbf{R}^n$. A twice continuously differentiable, complex-valued function u defined on Ω is *harmonic* on Ω if

$$\Delta u \equiv 0,$$

where $\Delta = D_1{}^2 + \cdots + D_n{}^2$ and $D_j{}^2$ denotes the second partial derivative with respect to the jth coordinate variable. The operator Δ is called the *Laplacian*, and the equation $\Delta u \equiv 0$ is called *Laplace's equation*. We say that a function u defined on a (not necessarily open) set $E \subset \mathbf{R}^n$ is harmonic on E if u can be extended to a function harmonic on an open set containing E.

We let $x = (x_1, \ldots, x_n)$ denote a typical point in $\mathbf{R}^n$ and let $|x| = (x_1{}^2 + \cdots + x_n{}^2)^{1/2}$ denote the Euclidean norm of x.

The simplest nonconstant harmonic functions are the coordinate functions; for example, $u(x) = x_1$. A slightly more complex example is the function on $\mathbf{R}^3$ defined by

$$u(x) = x_1{}^2 + x_2{}^2 - 2x_3{}^2 + ix_2.$$

As we will see later, the function

$$u(x) = |x|^{2-n}$$

is vital to harmonic function theory when $n > 2$; the reader should verify that this function is harmonic on $\mathbf{R}^n \setminus \{0\}$.

We can obtain additional examples of harmonic functions by differentiation, noting that for smooth functions the Laplacian commutes with any partial derivative. In particular, differentiating the last example with respect to x_1 shows that $x_1|x|^{-n}$ is harmonic on $\mathbf{R}^n \setminus \{0\}$ when $n > 2$. (We will soon prove that every harmonic function is infinitely differentiable; thus every partial derivative of a harmonic function is harmonic.)

The function $x_1|x|^{-n}$ is harmonic on $\mathbf{R}^n \setminus \{0\}$ even when $n = 2$. This can be verified directly or by noting that $x_1|x|^{-2}$ is a partial derivative of $\log|x|$, a harmonic function on $\mathbf{R}^2 \setminus \{0\}$. The function $\log|x|$ plays the same role when $n = 2$ that $|x|^{2-n}$ plays when $n > 2$. Notice that $\lim_{x\to\infty} \log|x| = \infty$, but $\lim_{x\to\infty} |x|^{2-n} = 0$; note also that $\log|x|$ is neither bounded above nor below, but $|x|^{2-n}$ is always positive. These facts hint at the contrast between harmonic function theory in the plane and in higher dimensions. Another key difference arises from the close connection between holomorphic and harmonic functions in the plane—a real-valued function on $\Omega \subset \mathbf{R}^2$ is harmonic if and only if it is locally the real part of a holomorphic function. No comparable result exists in higher dimensions.

Invariance Properties

Throughout this book, all functions are assumed to be complex valued unless stated otherwise. For k a positive integer, let $C^k(\Omega)$ denote the set of k times continuously differentiable functions on Ω; $C^\infty(\Omega)$ is the set of functions that belong to $C^k(\Omega)$ for every k. For $E \subset \mathbf{R}^n$, we let $C(E)$ denote the set of continuous functions on E.

Because the Laplacian is linear on $C^2(\Omega)$, sums and scalar multiples of harmonic functions are harmonic.

For $y \in \mathbf{R}^n$ and u a function on Ω, the *y-translate* of u is the function on $\Omega + y$ whose value at x is $u(x - y)$. Clearly, translations of harmonic functions are harmonic.

For a positive number r and u a function on Ω, the *r-dilate* of u, denoted u_r, is the function

$$(u_r)(x) = u(rx)$$

defined for x in $(1/r)\Omega = \{(1/r)w : w \in \Omega\}$. If $u \in C^2(\Omega)$, then a simple computation shows that $\Delta(u_r) = r^2(\Delta u)_r$ on $(1/r)\Omega$. Hence dilates of harmonic functions are harmonic.

Note the formal similarity between the Laplacian $\Delta = D_1{}^2 + \cdots + D_n{}^2$ and the function $|x|^2 = x_1{}^2 + \cdots + x_n{}^2$, whose level sets are spheres centered at the origin. The connection between harmonic functions and spheres is central to harmonic function theory. The mean-value property, which we discuss in the next section, best illustrates this connection. Another connection involves linear transformations on $\mathbf{R}^n$ that preserve the unit sphere; such transformations are called *orthogonal*. A linear map $T: \mathbf{R}^n \to \mathbf{R}^n$ is orthogonal if and only if $|Tx| = |x|$ for all $x \in \mathbf{R}^n$. Simple linear algebra shows that T is orthogonal if and only if the column vectors of the matrix of T (with respect to the standard basis of $\mathbf{R}^n$) form an orthonormal set.

We now show that the Laplacian commutes with orthogonal transformations; more precisely, if T is orthogonal and $u \in C^2(\Omega)$, then

$$\Delta(u \circ T) = (\Delta u) \circ T$$

on $T^{-1}(\Omega)$. To prove this, let $[t_{jk}]$ denote the matrix of T relative to the standard basis of $\mathbf{R}^n$. Then

$$D_m(u \circ T) = \sum_{j=1}^{n} t_{jm}(D_j u) \circ T,$$

where D_m denotes the partial derivative with respect to the mth coordinate variable. Differentiating once more and summing over m yields

$$\begin{aligned}
\Delta(u \circ T) &= \sum_{m=1}^{n} \sum_{j,k=1}^{n} t_{km} t_{jm} (D_k D_j u) \circ T \\
&= \sum_{j,k=1}^{n} \Big(\sum_{m=1}^{n} t_{km} t_{jm} \Big) (D_k D_j u) \circ T \\
&= \sum_{j=1}^{n} (D_j D_j u) \circ T \\
&= (\Delta u) \circ T,
\end{aligned}$$

as desired. The function $u \circ T$ is called a *rotation* of u. The preceding calculation shows that rotations of harmonic functions are harmonic.

The Mean-Value Property

Many basic properties of harmonic functions follow from Green's identity (which we will need mainly in the special case when Ω is a ball):

1.1
$$\int_\Omega (u\Delta v - v\Delta u)\,dV = \int_{\partial\Omega} (uD_{\mathbf{n}}v - vD_{\mathbf{n}}u)\,ds.$$

Here Ω is a bounded open subset of $\mathbf{R}^n$ with smooth boundary, and u and v are C^2-functions on a neighborhood of $\overline{\Omega}$, the closure of Ω. The measure $V = V_n$ is Lebesgue volume measure on $\mathbf{R}^n$, and s denotes surface-area measure on $\partial\Omega$ (see Appendix A for a discussion of integration over balls and spheres). The symbol $D_{\mathbf{n}}$ denotes differentiation with respect to the outward unit normal $\mathbf{n}$. Thus for $\zeta \in \partial\Omega$, $(D_{\mathbf{n}}u)(\zeta) = (\nabla u)(\zeta) \cdot \mathbf{n}(\zeta)$, where $\nabla u = (D_1 u, \dots, D_n u)$ denotes the gradient of u and $\cdot$ denotes the usual Euclidean inner product.

Green's identity (1.1) follows easily from the familiar divergence theorem of advanced calculus:

1.2
$$\int_\Omega \operatorname{div}\mathbf{w}\,dV = \int_{\partial\Omega} \mathbf{w}\cdot\mathbf{n}\,ds.$$

Here $\mathbf{w} = (w_1, \dots, w_n)$ is a smooth vector field (a $\mathbf{C}^n$-valued function whose components are continuously differentiable) on a neighborhood of $\overline{\Omega}$, and $\operatorname{div}\mathbf{w}$, the divergence of $\mathbf{w}$, is defined to be $D_1 w_1 + \cdots + D_n w_n$. To obtain Green's identity from the divergence theorem, simply let $\mathbf{w} = u\nabla v - v\nabla u$ and compute.

The following useful form of Green's identity occurs when u is harmonic and $v \equiv 1$:

1.3
$$\int_{\partial\Omega} D_{\mathbf{n}}u\,ds = 0.$$

Green's identity is the key to the proof of the mean-value property. Before stating the mean-value property, we introduce some notation: $B(a,r) = \{x \in \mathbf{R}^n : |x - a| < r\}$ is the open ball centered at a of

radius r; its closure is the closed ball $\overline{B}(a,r)$; the unit ball $B(0,1)$ is denoted by B and its closure by $\overline{B}$. When the dimension is important we write B_n in place of B. The unit sphere, the boundary of B, is denoted by S; normalized surface-area measure on S is denoted by σ (so that $\sigma(S) = 1$). The measure σ is the unique Borel probability measure on S that is rotation invariant (meaning $\sigma(T(E)) = \sigma(E)$ for every Borel set $E \subset S$ and every orthogonal transformation T).

1.4 Mean-Value Property: *If u is harmonic on $\overline{B}(a,r)$, then u equals the average of u over $\partial B(a,r)$. More precisely,*

$$u(a) = \int_S u(a + r\zeta)\, d\sigma(\zeta).$$

PROOF: First assume that $n > 2$. Without loss of generality we may assume that $B(a,r) = B$. Fix $\varepsilon \in (0,1)$. Apply Green's identity (1.1) with $\Omega = \{x \in \mathbf{R}^n : \varepsilon < |x| < 1\}$ and $v(x) = |x|^{2-n}$ to obtain

$$\begin{aligned} 0 = (2-n)\int_S u\, ds - (2-n)\varepsilon^{1-n}\int_{\varepsilon S} u\, ds \\ - \int_S D_{\mathbf{n}} u\, ds - \varepsilon^{2-n}\int_{\varepsilon S} D_{\mathbf{n}} u\, ds. \end{aligned}$$

By 1.3, the last two terms are 0, thus

$$\int_S u\, ds = \varepsilon^{1-n}\int_{\varepsilon S} u\, ds,$$

which is the same as

$$\int_S u\, d\sigma = \int_S u(\varepsilon\zeta)\, d\sigma(\zeta).$$

Letting $\varepsilon \to 0$ and using the continuity of u at 0, we obtain the desired result.

The proof when $n = 2$ is the same, except that $|x|^{2-n}$ should be replaced by $\log|x|$. ∎

Harmonic functions also have a mean-value property with respect to volume measure. The polar coordinates formula for integration on $\mathbf{R}^n$ is indispensable here. The formula states that for a Borel measurable, integrable function f on $\mathbf{R}^n$,

1.5 $$\frac{1}{nV(B)}\int_{\mathbf{R}^n} f\,dV = \int_0^\infty r^{n-1}\int_S f(r\zeta)\,d\sigma(\zeta)\,dr$$

(see [15], Chapter 8, Exercise 6). The constant $nV(B)$ arises from the normalization of σ (choosing f to be the characteristic function of B shows that $nV(B)$ is the correct constant).

1.6 Mean-Value Property, Volume Version: *If u is harmonic on $\overline{B}(a,r)$, then $u(a)$ equals the average of u over $B(a,r)$. More precisely,*

$$u(a) = \frac{1}{V(B(a,r))}\int_{B(a,r)} u\,dV.$$

PROOF: We can assume that $B(a,r) = B$. Apply the polar coordinates formula (1.5) with f equal to u times the characteristic function of B, and then use the spherical mean-value property (Theorem 1.4). ∎

We will see later (1.24 and 1.25) that the mean-value property characterizes harmonic functions.

We conclude this section with an application of the mean value property. We have seen that a real-valued harmonic function may have an isolated (nonremovable) singularity; for example, $|x|^{2-n}$ has an isolated singularity at 0 if $n > 2$. However, a real-valued harmonic function u cannot have isolated zeros.

1.7 Corollary: *The zeros of a real-valued harmonic function are never isolated.*

PROOF: Suppose u is harmonic and real valued on Ω, $a \in \Omega$, and $u(a) = 0$. Let $r > 0$ be such that $\overline{B}(a,r) \subset \Omega$. Because the average of u over $\partial B(a,r)$ equals 0, either u is identically 0 on $\partial B(a,r)$ or u takes on both positive and negative values on $\partial B(a,r)$. In the later case, the connectedness of $\partial B(a,r)$ implies that u has a zero on $\partial B(a,r)$.

Thus u has a zero on the boundary of every sufficiently small ball centered at a, proving that a is not an isolated zero of u. ∎

The hypothesis that u is real valued is needed in the preceding corollary. This is no surprise when $n = 2$, because nonconstant holomorphic functions have isolated zeros. When $n \geq 2$, the harmonic function

$$(1-n)x_1{}^2 + \sum_{k=2}^{n} x_k{}^2 + ix_1$$

is an example; it vanishes only at the origin.

The Maximum Principle

An important consequence of the mean-value property is the following maximum principle for harmonic functions.

1.8 Maximum Principle: *Suppose Ω is connected, u is real valued and harmonic on Ω, and u has a maximum or a minimum in Ω. Then u is constant.*

PROOF: Suppose u attains a maximum at $a \in \Omega$. Choose $r > 0$ such that $\overline{B}(a,r) \subset \Omega$. If u were less than $u(a)$ at some point of $B(a,r)$, then the continuity of u would show that the average of u over $B(a,r)$ is less than $u(a)$, contradicting 1.6. Therefore u is constant on $B(a,r)$, proving that the set where u attains its maximum is open in Ω. Because this set is also closed in Ω (again by the continuity of u), it must be all of Ω (by connectivity). Thus u is constant on Ω, as desired.

If u attains a minimum in Ω, we can apply this argument to $-u$. ■

The following corollary, whose proof immediately follows from the preceding theorem, is frequently useful. (Note that the connectivity of Ω is not needed here.)

1.9 Corollary: *Suppose Ω is bounded and u is a continuous real-valued function on $\overline{\Omega}$ that is harmonic on Ω. Then u attains its maximum and minimum values over $\overline{\Omega}$ on $\partial\Omega$.*

The corollary above implies that on a bounded domain a harmonic function is determined by its boundary values. More precisely, for bounded Ω, if u and v are continuous functions on $\overline{\Omega}$ that are harmonic on Ω, and if $u = v$ on $\partial\Omega$, then $u = v$ on Ω. Unfortunately this can fail on an unbounded domain. For example, the harmonic functions $u(x) = 0$ and $v(x) = x_n$ agree on the boundary of the half-space $\{x \in \mathbf{R}^n : x_n > 0\}$.

The next version of the maximum principle can be applied even when Ω is unbounded or when u is not continuous on $\overline{\Omega}$.

1.10 Corollary: *Let u be a real-valued, harmonic function on Ω, and suppose*

$$\limsup_{k \to \infty} u(a_k) \le M$$

for every sequence (a_k) in Ω converging either to a point in $\partial\Omega$ or to ∞. Then $u \le M$ on Ω.

REMARK: To say that (a_k) converges to ∞ means that $|a_k| \to \infty$. The corollary is valid if "lim sup" is replaced by "lim inf" and the inequalities are reversed.

PROOF OF COROLLARY 1.10: Let $M' = \sup\{u(x) : x \in \Omega\}$, and choose a sequence (b_k) in Ω such that $u(b_k) \to M'$.

If (b_k) has a subsequence converging to some point $b \in \Omega$, then $u(b) = M'$, which implies u is constant on the component of Ω containing b (by the maximum principle). Hence in this case there is a sequence (a_k) in Ω converging to a boundary point of Ω or to ∞ on which $u = M'$, and so $M' \le M$.

If no subsequence of (b_k) converges to a point in Ω, then (b_k) has a subsequence (a_k) converging either to a boundary point of Ω or to ∞. Thus in in this case we also have $M' \le M$. ■

Theorem 1.8 and Corollaries 1.9 and 1.10 apply only to real-valued functions. The next corollary is a version of the maximum principle for complex-valued functions.

1.11 Corollary: *Let Ω be connected, and let u be harmonic on Ω. If $|u|$ has a maximum in Ω, then u is constant.*

PROOF: Suppose $|u|$ attains a maximum value of M at some point $a \in \Omega$. Choose $\lambda \in \mathbf{C}$ such that $|\lambda| = 1$ and $\lambda u(a) = M$. Then the real-valued harmonic function $\operatorname{Re} \lambda u$ attains its maximum value M at a; thus by Theorem 1.8, $\operatorname{Re} \lambda u \equiv M$ on Ω. Because $|\lambda u| = |u| \le M$, we have $\operatorname{Im} \lambda u \equiv 0$ on Ω. Thus λu, and hence u, is constant on Ω. ■

Corollary 1.11 is the analogue of Theorem 1.8 for complex-valued harmonic functions; the corresponding analogues of Corollaries 1.9

and 1.10 are also valid. All these analogues, however, hold only for the maximum or lim sup of $|u|$. No minimum principle holds for $|u|$ (consider $u(x) = x_1$ on B).

We will be able to prove a local version of the maximum principle after we prove that harmonic functions are real analytic (see 1.29).

The Poisson Kernel for the Ball

The mean-value property shows that if u is harmonic on $\overline{B}$, then

$$u(0) = \int_S u(\zeta)\, d\sigma(\zeta).$$

We now show that for every $x \in B$, $u(x)$ is a weighted average of u over S. More precisely, we will show there exists a function P on $B \times S$ such that

$$u(x) = \int_S u(\zeta)P(x, \zeta)\, d\sigma(\zeta)$$

for every $x \in B$ and every u harmonic on $\overline{B}$.

To discover what P might be, we start with the special case $n = 2$. Suppose u is a real-valued harmonic function on the closed unit disk in $\mathbf{R}^2$. Then $u = \operatorname{Re} f$ for some function f holomorphic on a neighborhood of the closed disk (see Exercise 11 of this chapter). Because $u = (f + \overline{f})/2$, the Taylor series expansion of f implies that u has the form

$$u(r\zeta) = \sum_{j=-\infty}^{\infty} a_j r^{|j|} \zeta^j,$$

where $0 \le r \le 1$ and $|\zeta| = 1$. In this formula, take $r = 1$, multiply both sides by ζ^{-k}, then integrate over the unit circle to obtain

$$a_k = \int_S u(\zeta)\zeta^{-k}\, d\sigma(\zeta).$$

Now let x be a point in the open unit disk, and write $x = r\eta$ with $r \in [0, 1)$ and $|\eta| = 1$. Then

1.12

$$\begin{aligned} u(x) &= u(r\eta) \\ &= \sum_{j=-\infty}^{\infty} \Big(\int_S u(\zeta)\zeta^{-j}\, d\sigma(\zeta) \Big) r^{|j|}\eta^j \\ &= \int_S u(\zeta) \Big(\sum_{j=-\infty}^{\infty} r^{|j|}(\eta\zeta^{-1})^j \Big)\, d\sigma(\zeta). \end{aligned}$$

Breaking the last sum into two geometric series, we see that

$$u(x) = \int_S u(\zeta) \frac{1-r^2}{|r\eta - \zeta|^2}\, d\sigma(\zeta).$$

Thus, letting $P(x,\zeta) = (1-|x|^2)/|x-\zeta|^2$, we obtain the desired formula for $n = 2$:

$$u(x) = \int_S u(\zeta)P(x,\zeta)\, d\sigma(\zeta).$$

Unfortunately, nothing as simple as this works in higher dimensions. To find $P(x,\zeta)$ when $n > 2$, we start with a result we call the symmetry lemma, which will be useful in other contexts as well.

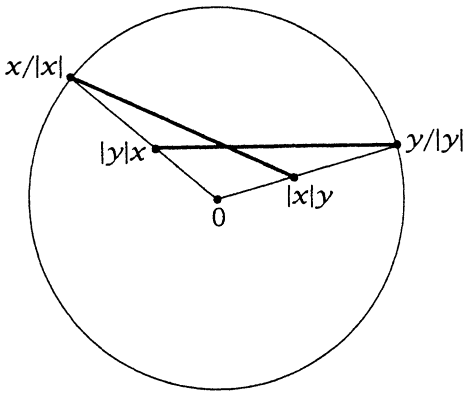
1.13 Symmetry Lemma: *For all nonzero x and y in $\mathbf{R}^n$,*

$$\Big| \frac{y}{|y|} - |y|x \Big| = \Big| \frac{x}{|x|} - |x|y \Big|.$$

PROOF: Square both sides and expand using the inner product. ∎

To find P for $n > 2$, we try the same approach used in proving the mean-value property. Suppose that u is harmonic on $\overline{B}$. When proving that $u(0)$ is the average of u over S, we applied Green's identity with $v(y) = |y|^{2-n}$; this function is harmonic on $B \setminus \{0\}$, has a singularity at 0, and is constant on S. Now fix a nonzero point $x \in B$. To show that $u(x)$ is a weighted average of u over S, it is natural this time to try $v(y) = |y-x|^{2-n}$. This function is harmonic on $B \setminus \{x\}$, has a singularity at x, but unfortunately is not constant on S. However, the symmetry lemma (1.13) shows that for $y \in S$,

$$|y-x|^{2-n} = |x|^{2-n} \Big| y - \frac{x}{|x|^2} \Big|^{2-n}.$$



The symmetry lemma: the two bold segments have the same length.

Notice that the right side of this equation is harmonic (as a function of y) on $\overline{B}$. Thus the difference of the left and right sides has all the properties we seek.

So set $v(y) = \mathcal{L}(y) - \mathcal{R}(y)$, where

$$\mathcal{L}(y) = |y - x|^{2-n}, \qquad \mathcal{R}(y) = |x|^{2-n}\left|y - \frac{x}{|x|^2}\right|^{2-n},$$

and choose ε small enough so that $\overline{B}(x, \varepsilon) \subset B$. Now apply Green's identity (1.1) much as in the proof of the mean-value property (1.4), with $\Omega = B \setminus \overline{B}(x, \varepsilon)$. We obtain

$$0 = \int_S u D_{\mathbf{n}} v \, ds - (2 - n)s(S)u(x)$$
$$- \int_{\partial B(x,\varepsilon)} u D_{\mathbf{n}} \mathcal{R} \, ds + \int_{\partial B(x,\varepsilon)} \mathcal{R} D_{\mathbf{n}} u \, ds$$

(the mean-value property was used here). Because $u D_{\mathbf{n}} \mathcal{R}$ and $\mathcal{R} D_{\mathbf{n}} u$ are bounded on B, the last two terms approach 0 as $\varepsilon \to 0$. Hence

$$u(x) = \frac{1}{2 - n} \int_S u D_{\mathbf{n}} v \, d\sigma.$$

Setting $P(x, \zeta) = (2 - n)^{-1}(D_{\mathbf{n}} v)(\zeta)$, we have the desired formula:

1.14 $$u(x) = \int_S u(\zeta) P(x, \zeta) \, d\sigma(\zeta).$$

A computation of $D_{\mathbf{n}}v$, which we recommend to the reader (the symmetry lemma may be useful here), yields

1.15
$$P(x,\zeta) = \frac{1-|x|^2}{|x-\zeta|^n}.$$

The function P derived above is called the *Poisson kernel* for the ball; it plays a key role in the next section.

The Dirichlet Problem for the Ball

We now come to a famous problem in harmonic function theory: given a continuous function f on S, does there exist a continuous function u on $\overline{B}$, with u harmonic on B, such that $u = f$ on S? If so, how do we find u? This is the *Dirichlet problem* for the ball. Recall that by the maximum principle, if a solution exists, then it is unique.

We take our cue from the last section. If f happens to be the restriction to S of a function u harmonic on $\overline{B}$, then

$$u(x) = \int_S f(\zeta)P(x,\zeta)\,d\sigma(\zeta)$$

for all $x \in B$. We solve the Dirichlet problem for B by changing our perspective. Starting with a continuous function f on S, we use the formula above to define an extension of f into B that we hope will have the desired properties.

The reader who wishes may regard the material in the last section as motivational. We now start anew, using 1.15 as the definition of $P(x,\zeta)$.

For arbitrary $f \in C(S)$, we define the *Poisson integral* of f, denoted $P[f]$, to be the function on B given by

1.16
$$P[f](x) = \int_S f(\zeta)P(x,\zeta)\,d\sigma(\zeta).$$

The next theorem shows that the Poisson integral solves the Dirichlet problem for B.



Johann Peter Gustav Lejeune Dirichlet (1805–1859), whose attempt to prove the stability of the solar system led to an investigation of harmonic functions.

1.17 Solution of the Dirichlet problem for the ball: *Suppose f is continuous on S. Define u on $\overline{B}$ by*

$$u(x) = \begin{cases} P[f](x) & \text{if } x \in B \\ f(x) & \text{if } x \in S. \end{cases}$$

Then u is continuous on $\overline{B}$ and harmonic on B.

The proof of 1.17 depends on harmonicity and approximate-identity properties of the Poisson kernel given in the following two propositions.

1.18 Proposition: *Let $\zeta \in S$. Then $P(\cdot, \zeta)$ is harmonic on $\mathbf{R}^n \setminus \{\zeta\}$.*

We let the reader prove this proposition. One way to do so is to write $P(x, \zeta) = (1 - |x|^2)|x - \zeta|^{-n}$ and then compute the Laplacian of $P(\cdot, \zeta)$ using the product rule

1.19 $$\Delta(uv) = u\Delta v + 2\nabla u \cdot \nabla v + v\Delta u,$$

which is valid for all real-valued twice continuously differentiable functions u and v.

1.20 Proposition: *The Poisson kernel has the following properties:*

(a) $P(x, \zeta) > 0$ *for all* $x \in B$ *and all* $\zeta \in S$;

(b) $\int_S P(x, \zeta)\, d\sigma(\zeta) = 1$ *for all* $x \in B$;

(c) *for every* $\eta \in S$ *and every* $\delta > 0$,

$$\int_{|\zeta - \eta| > \delta} P(x, \zeta)\, d\sigma(\zeta) \to 0 \quad \text{as } x \to \eta.$$

PROOF: Properties (a) and (c) follow immediately from the formula for the Poisson kernel (1.15).

Taking u to be identically 1 in 1.14 gives (b). To prove (b) without using the motivational material in the last section, note that for $x \in B \setminus \{0\}$, we have

$$\begin{aligned}\int_S P(x, \zeta)\, d\sigma(\zeta) &= \int_S P(|\zeta| x, \frac{\zeta}{|\zeta|})\, d\sigma(\zeta) \\ &= \int_S P(|x|\zeta, \frac{x}{|x|})\, d\sigma(\zeta),\end{aligned}$$

where the last equality follows from the symmetry lemma (1.13). Proposition 1.18 tells us that $P(|x|\zeta, \frac{x}{|x|})$, as a function of ζ, is harmonic on $\overline{B}$. Thus by the mean-value property we have

$$\int_S P(x, \zeta)\, d\sigma(\zeta) = P(0, \frac{x}{|x|}) = 1,$$

as desired. Clearly (b) also holds for $x = 0$, completing the proof. ∎

PROOF OF THEOREM 1.17: The Laplacian of u can be computed by differentiating under the integral sign in 1.16; Proposition 1.18 then shows that u is harmonic on B.

To prove that u is continuous on $\overline{B}$, fix $\eta \in S$ and $\varepsilon > 0$. Choose $\delta > 0$ such that $|f(\zeta) - f(\eta)| < \varepsilon$ whenever $|\zeta - \eta| < \delta$ (and $\zeta \in S$). For $x \in B$, (a) and (b) of Proposition 1.20 imply that

$$
\begin{aligned}
|u(x) - u(\eta)| &= \left| \int_S (f(\zeta) - f(\eta)) P(x, \zeta)\, d\sigma(\zeta) \right| \\
&\le \int_{|\zeta - \eta| \le \delta} |f(\zeta) - f(\eta)|\, P(x, \zeta)\, d\sigma(\zeta) \\
&\qquad + \int_{|\zeta - \eta| > \delta} |f(\zeta) - f(\eta)|\, P(x, \zeta)\, d\sigma(\zeta) \\
&\le \varepsilon + 2\|f\|_\infty \int_{|\zeta - \eta| > \delta} P(x, \zeta)\, d\sigma(\zeta),
\end{aligned}
$$

where $\|f\|_\infty$ denotes the supremum of $|f|$ on S. The last term above is less than ε for x sufficiently close to η (by Proposition 1.20(c)), proving that u is continuous at η. ∎

We now prove a result stronger than that expressed in 1.14.

1.21 Theorem: *If u is a continuous function on $\overline{B}$ that is harmonic on B, then $u = P[u|_S]$ on B.*

PROOF: By 1.17, $u - P[u|_S]$ is harmonic on B and extends continuously to be 0 on S. The maximum principle (Corollary 1.9) now implies that $u - P[u|_S]$ is 0 on B. ∎

Because translations and dilations preserve harmonic functions, our results can be transferred easily to any ball $B(a, r)$. Specifically, given a continuous function f on $\partial B(a, r)$, there exists a unique continuous function u on $\overline{B}(a, r)$, with u harmonic on $B(a, r)$, such that $u = f$ on $\partial B(a, r)$. In this case we say that u solves the Dirichlet problem for $B(a, r)$ with boundary data f.

We now show that every harmonic function is infinitely differentiable. In dealing with differentiation in several variables the following notation is useful: a *multi-index* α is an n-tuple of nonnegative integers $(\alpha_1, \ldots, \alpha_n)$; the partial differentiation operator D^α is defined to be $D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ (D_j^0 denotes the identity operator). For each $\zeta \in S$, the function $P(\cdot, \zeta)$ is infinitely differentiable on B; we denote its αth partial derivative by $D^\alpha P(\cdot, \zeta)$ (here ζ is held fixed).

If u is continuous on $\overline{B}$ and harmonic on B, then

$$
u(x) = \int_S u(\zeta) P(x, \zeta)\, d\sigma(\zeta)
$$

for every $x \in B$. Differentiating under the integral, we easily see that $u \in C^\infty(B)$; the formula

1.22
$$D^\alpha u(x) = \int_S u(\zeta) D^\alpha P(x,\zeta)\, d\sigma(\zeta)$$

holds for every $x \in B$ and every multi-index α.

The preceding argument applies to any ball after a translation and dilation. As a consequence, every harmonic function is infinitely differentiable.

The following theorem should remind the reader of the behavior of a uniformly convergent sequence of holomorphic functions.

1.23 Theorem: *Suppose (u_m) is a sequence of harmonic functions on Ω such that u_m converges uniformly to a function u on each compact subset of Ω. Then u is harmonic on Ω. Moreover, for every multi-index α, $D^\alpha u_m$ converges uniformly to $D^\alpha u$ on each compact subset of Ω.*

PROOF: Given $\overline{B}(a,r) \subset \Omega$, we need only show that u is harmonic on $B(a,r)$ and that for every multi-index α, $D^\alpha u_m$ converges uniformly to $D^\alpha u$ on each compact subset of $B(a,r)$. Without loss of generality, we assume $B(a,r) = B$.

We then know that

$$u_m(x) = \int_S u_m(\zeta) P(x,\zeta)\, d\sigma(\zeta)$$

for every $x \in B$ and every m. Taking the limit of both sides, we obtain

$$u(x) = \int_S u(\zeta) P(x,\zeta)\, d\sigma(\zeta)$$

for every $x \in B$. Thus u is harmonic on B.

Let α be a multi-index and let $x \in B$. Then

$$\begin{aligned} D^\alpha u_m(x) &= \int_S u_m(\zeta) D^\alpha P(x,\zeta)\, d\sigma(\zeta) \\ &\to \int_S u(\zeta) D^\alpha P(x,\zeta)\, d\sigma(\zeta) = D^\alpha u(x). \end{aligned}$$

If K is a compact subset of B, then $D^\alpha P$ is uniformly bounded on $K \times S$, and so the convergence of $D^\alpha u_m$ to $D^\alpha u$ is uniform on K, as desired. ∎

Converse of the Mean-Value Property

We have seen that every harmonic function has the mean-value property. In this section, we use the solvability of the Dirichlet problem for the ball to prove that harmonic functions are the only continuous functions having the mean-value property. In fact, the following theorem shows that a continuous function satisfying a weak form of the mean-value property must be harmonic.

1.24 Theorem: *Suppose u is a continuous function on Ω. If for each $x \in \Omega$ there is a sequence of positive numbers $r_j \to 0$ such that*

$$u(x) = \int_S u(x + r_j\zeta)\, d\sigma(\zeta)$$

for all j, then u is harmonic on Ω.

PROOF: Without loss of generality, we can assume that u is real valued. Suppose that $\overline{B}(a,R) \subset \Omega$. Let v solve the Dirichlet problem for $\overline{B}(a,R)$ with boundary data u on $\partial B(a,R)$. We will complete the proof by showing that $v = u$ on $B(a,R)$.

Suppose that $v - u$ is positive at some point of $\overline{B}(a,R)$. Let E be the subset of $\overline{B}(a,R)$ where $v - u$ attains its maximum. Because E is compact, E contains a point x farthest from a. Clearly $x \in B(a,R)$, so there exists a ball $B(x,r) \subset B(a,R)$ such that $u(x)$ equals the average of u over $\partial B(x,r)$.

Because v is harmonic, we have

$$(v - u)(x) = \int_S (v - u)(x + r\zeta)\, d\sigma(\zeta).$$

But $(v - u)(x + r\zeta) \leq (v - u)(x)$ for all $\zeta \in S$, with strict inequality on a nonempty open subset of S (because of how x was chosen), contradicting the equation above. Thus $v - u \leq 0$ on $\overline{B}(a,R)$. Similarly, $v - u \geq 0$ on $\overline{B}(a,R)$. ∎

The proof above can be modified to show that if u is continuous on Ω and satisfies a local mean-value property with respect to volume measure, then u is harmonic on Ω; see Exercise 22 of this chapter.

The hypothesis of continuity is needed in Theorem 1.24. To see this, let $\Omega = \mathbf{R}^n$ and define u by

$$u(x) = \begin{cases} 1 & \text{if } x_n > 0 \\ 0 & \text{if } x_n = 0 \\ -1 & \text{if } x_n < 0. \end{cases}$$

Then $u(x)$ equals the average of u over every sphere centered at x if $x_n = 0$, and $u(x)$ equals the average of u over all sufficiently small spheres centered at x if $x_n \neq 0$. But u is not even continuous, much less harmonic, on $\mathbf{R}^n$.

In the following theorem we replace the continuity assumption with the weaker condition of local integrability (a function is *locally integrable* on Ω if it is Lebesgue integrable on every compact subset of Ω). However, we now require that the averaging property (with respect to volume measure) hold for every radius.

1.25 Theorem: *If u is a locally integrable function on Ω such that*

$$u(a) = \frac{1}{V(B(a,r))} \int_{B(a,r)} u\, dV$$

whenever $\overline{B}(a,r) \subset \Omega$, then u is harmonic on Ω.

PROOF: By Exercise 22 of this chapter, we need only show that u is continuous on Ω. Fix $a \in \Omega$ and let (a_j) be a sequence in Ω converging to a. Let K be a compact subset of Ω with a in the interior of K. Then there exists an $r > 0$ such that $B(a_j, r) \subset K$ for all sufficiently large j. Because u is integrable on K, the dominated convergence theorem shows that

$$\begin{aligned} u(a_j) &= \frac{1}{V(B(a,r))} \int_{B(a_j,r)} u\, dV \\ &= \frac{1}{V(B(a,r))} \int_K u\chi_{B(a_j,r)}\, dV \\ &\to \frac{1}{V(B(a,r))} \int_K u\chi_{B(a,r)}\, dV = u(a) \end{aligned}$$

(as usual, χ_E denotes the function that is 1 on E and 0 off E). Thus u is continuous on Ω, as desired. ∎

Real Analyticity and Homogeneous Expansions

We saw in the section before last that harmonic functions are infinitely differentiable. A much stronger property will be established in this section—harmonic functions are real analytic. Roughly speaking, a function is real analytic if it is locally expressible as a power series in the coordinate variables $x_1, x_2, \ldots, x_n$ of $\mathbf{R}^n$.

To make this more precise, we need to discuss what is meant by a series of complex numbers of the form $\sum c_\alpha$, where the summation is over all multi-indices α. (The full range of multi-indices will always be intended in a series unless indicated otherwise.) The problem is that there is no natural ordering of the set of all multi-indices when $n > 1$. However, suppose we know that $\sum c_\alpha$ is absolutely convergent, i.e., that

$$\sup \sum_{\alpha \in F} |c_\alpha| < \infty,$$

where the supremum is taken over all finite subsets F of multi-indices. All orderings $\alpha(1), \alpha(2), \ldots$ of multi-indices then yield the same value for $\sum_{j=1}^{\infty} c_{\alpha(j)}$; hence we may unambiguously write $\sum c_\alpha$ for this value. We will only be concerned with such absolutely convergent series.

The following notation will be convenient when dealing with multiple power series: for $x \in \mathbf{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ a multi-index, define

$$x^\alpha = {x_1}^{\alpha_1} {x_2}^{\alpha_2} \ldots {x_n}^{\alpha_n},$$

$$\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!,$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

A function f on Ω is *real analytic* on Ω if for every $a \in \Omega$ there exist complex numbers c_α such that

$$f(x) = \sum c_\alpha (x - a)^\alpha$$

for all x in a neighborhood of a, the series converging absolutely in this neighborhood.

Some basic properties of such series are contained in the next proposition. Here it will be convenient to center the power series at $a = 0$, and to define

$$R(y) = \{x \in \mathbf{R}^n : |x_j| < |y_j|, \quad j = 1, 2, \dots, n\}$$

for $y \in \mathbf{R}^n$; $R(y)$ is the n-dimensional open rectangle centered at 0 with "corner y". To avoid trivialities we will assume that each component of y is nonzero.

1.26 Theorem: *Suppose $\{c_\alpha y^\alpha\}$ is a bounded set. Then:*

(a) *For every multi-index β, the series*

$$\sum_\alpha D^\beta (c_\alpha x^\alpha)$$

converges absolutely on $R(y)$ and uniformly on compact subsets of $R(y)$.

(b) *The function f defined by $f(x) = \sum c_\alpha x^\alpha$ for $x \in R(y)$ is infinitely differentiable on $R(y)$. Moreover,*

$$D^\beta f(x) = \sum_\alpha D^\beta (c_\alpha x^\alpha)$$

for all $x \in R(y)$ and for every multi-index β. Furthermore, $c_\alpha = D^\alpha f(0)/\alpha!$ for every multi-index α.

REMARKS: 1. To say the preceding series converges uniformly on a set means that every ordering of the series converges uniformly on this set in the usual sense.

2. The theorem shows that every derivative of a real-analytic function is real analytic, and that if $\sum a_\alpha x^\alpha = \sum b_\alpha x^\alpha$ for all x in a neighborhood of 0, then $a_\alpha = b_\alpha$ for all α.

PROOF OF THEOREM 1.26: We first observe that on the rectangle $R((1, 1, \dots, 1))$, we have

$$\sum_\alpha D^\beta (x^\alpha) = D^\beta [(1 - x_1)^{-1} (1 - x_2)^{-1} \dots (1 - x_n)^{-1}]$$

for every multi-index β, as the reader should verify (start with $\beta = 0$).

Now assume that $|c_\alpha y^\alpha| \le M$ for every α. If K is a compact subset of $R(y)$, then $K \subset R(ty)$ for some $t \in (0,1)$. Thus for every $x \in K$ and every multi-index α,

$$|c_\alpha x^\alpha| \le t^{|\alpha|}|c_\alpha y^\alpha| \le Mt^{|\alpha|}.$$

By the preceding paragraph, $\sum t^{|\alpha|} = (1-t)^{-n} < \infty$, establishing the absolute and uniform convergence of $\sum c_\alpha x^\alpha$ on K. Similar reasoning, with a little more bookkeeping, applies to $\sum D^\beta(c_\alpha x^\alpha)$. This completes the proof of (a).

Letting $f(x) = \sum c_\alpha x^\alpha$ for $x \in R(y)$, the uniform convergence on compact subsets of $R(y)$ of the series $\sum D^\beta(c_\alpha x^\alpha)$ for every β shows that $f \in C^\infty(R(y))$, and that $D^\beta f(x) = \sum D^\beta(c_\alpha x^\alpha)$ in $R(y)$ for every β. The formula for the Taylor coefficients c_α follows from this by computing the derivatives of f at 0. ■

A word of caution: Theorem 1.26 does not assert that rectangles are the natural domains of convergence of multiple power series. For example, in two dimensions the domain of convergence of $\sum_{j=1}^\infty (x_1x_2)^j$ is $\{(x_1,x_2) \in \mathbf{R}^2 : |x_1x_2| < 1\}$.

The next theorem shows that real-analytic functions enjoy certain properties not shared by all C^∞-functions.

1.27 Theorem: *Suppose Ω is connected, f is real analytic in Ω, and $f = 0$ on a nonempty open subset of Ω. Then $f \equiv 0$ in Ω.*

PROOF: Let ω denote the interior of $\{x \in \Omega : f(x) = 0\}$. Then ω is an open subset of Ω. If $a \in \Omega$ is a limit point of ω, then all derivatives of f vanish at a by continuity, implying that the power series of f at a is identically zero; hence $a \in \omega$. Thus ω is closed in Ω. Because ω is nonempty by hypothesis, we must have $\omega = \Omega$ by connectivity, giving $f \equiv 0$ in Ω. ■

1.28 Theorem: *If u is harmonic on Ω, then u is real analytic in Ω.*

PROOF: It suffices to show that if u is harmonic on $\overline{B}$, then u has a power series expansion converging to u in a neighborhood of 0.

The main idea here is the same as in one complex variable—we use the Poisson integral representation of u and expand the Poisson kernel

in a power series. Unfortunately the details are not as simple as in the case of the Cauchy integral formula.

Suppose that $|x| < \sqrt{2} - 1$ and $\zeta \in S$. Then $0 < |x - \zeta|^2 < 2$, and thus

$$P(x, \zeta) = (1 - |x|^2)(|x - \zeta|^2)^{-n/2} = (1 - |x|^2) \sum_{m=0}^{\infty} c_m (|x|^2 - 2x \cdot \zeta)^m,$$

where $\sum_{m=0}^{\infty} c_m (t-1)^m$ is the Taylor series of $t^{-n/2}$ on the interval $(0, 2)$, expanded about the midpoint 1. After expanding the terms $(|x|^2 - 2x \cdot \zeta)^m$ and rearranging (permissible, since we have all of the absolute convergence one could ask for), the Poisson kernel takes the form

$$P(x, \zeta) = \sum_{\alpha} x^{\alpha} q_{\alpha}(\zeta),$$

for $x \in (\sqrt{2} - 1)B$ and $\zeta \in S$, where each q_α is a polynomial. This latter series converges uniformly on S for each $x \in (\sqrt{2} - 1)B$.

Thus if u is harmonic on $\overline{B}$,

$$\begin{aligned} u(x) &= \int_S u(\zeta) P(x, \zeta)\, d\sigma(\zeta) \\ &= \sum_{\alpha} \Big(\int_S u q_\alpha \, d\sigma \Big) x^{\alpha} \end{aligned}$$

for all $x \in (\sqrt{2} - 1)B$. This is the desired expansion of u near 0. ∎

Unfortunately, the multiple power series at 0 of a function harmonic on B need not converge in all of B. For example, the function $u(z) = 1/(1 - z)$ is holomorphic (hence harmonic) on the open unit disk of the complex plane. Writing $z = x + iy = (x, y) \in \mathbf{R}^2$, we have

$$u(z) = \sum_{m=0}^{\infty} (x + iy)^m = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \binom{m}{j} x^j (iy)^{m-j}$$

for $z \in B_2$. As a multiple power series, the last sum above converges absolutely if and only if $|x| + |y| < 1$, and hence does not converge in all of B_2. The reader should perhaps take a moment to meditate on the difference between the "real-analytic" and "holomorphic" power series of u.

As mentioned earlier, the real analyticity of harmonic functions allows us to prove a local maximum principle.

1.29 Local Maximum Principle: *Suppose Ω is connected, u is real valued and harmonic on Ω, and u has a local maximum in Ω. Then u is constant.*

PROOF: If u has a local maximum at $a \in \Omega$, then there exists a ball $B(a,r) \subset \Omega$ such that $u \le u(a)$ in $B(a,r)$. By Theorem 1.8, u is constant on $B(a,r)$. Because u is real analytic on Ω, $u \equiv u(a)$ in Ω by Theorem 1.27. ∎

Knowing that harmonic functions locally have power series expansions enables us to express them locally as sums of homogeneous harmonic polynomials. This has many interesting consequences, as we will see later. In the remainder of this section we develop a few basic results, starting with a brief discussion of homogeneous polynomials.

A polynomial is by definition a finite linear combination of monomials x^α. A polynomial p of the form

$$p(x) = \sum_{|\alpha|=m} c_\alpha x^\alpha$$

is said to be *homogeneous* of *degree m*; here we allow m to be any nonnegative integer. Equivalently, a polynomial p is homogeneous of degree m if

$$p(tx) = t^m p(x)$$

for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}^n$. This last formulation shows that a homogeneous polynomial is determined by its restriction to S: if p and q are homogeneous of degree m and $p = q$ on S, then $p = q$ on $\mathbf{R}^n$. (This is not true of polynomials in general; for example, $1 - |x|^2 \equiv 0$ on S.) Note also that if p is a homogeneous polynomial of degree m, then so is $p \circ T$ for every linear map T from $\mathbf{R}^n$ to $\mathbf{R}^n$.

It is often useful to express functions as infinite sums of homogeneous polynomials. Here is a simple uniqueness result for such sums.

1.30 Proposition: *Let $r > 0$. If p_m and q_m are homogeneous polynomials of degree m, $m = 0, 1, \dots$, and if*

$$\sum_{m=0}^{\infty} p_m(x) = \sum_{m=0}^{\infty} q_m(x)$$

for all $x \in rB$ (both series converging pointwise in rB), then $p_m = q_m$ for every m.

PROOF: Fix $\zeta \in S$. Since the two series above converge and are equal at each point in rB, we have

$$\sum_{m=0}^{\infty} p_m(\zeta)t^m = \sum_{m=0}^{\infty} q_m(\zeta)t^m$$

for all $t \in (-r, r)$. By the uniqueness of coefficients of power series in one variable, $p_m(\zeta) = q_m(\zeta)$ for every m. This is true for every $\zeta \in S$, and thus $p_m = q_m$ on S for all m. By the preceding remarks, $p_m = q_m$ on $\mathbf{R}^n$ for every m. ■

Suppose now that u is harmonic near 0. Letting

$$p_m(x) = \sum_{|\alpha|=m} \frac{D_\alpha u(0)}{\alpha!} x^\alpha,$$

we see from Theorem 1.28 that

$$u(x) = \sum_{m=0}^{\infty} p_m(x)$$

for x near 0. Because each p_m is homogeneous of degree m, the latter series is called the *homogeneous expansion* of u at 0. Remarkably, the harmonicity of u implies that each p_m is harmonic. To see this, observe that $\Delta u = \sum \Delta p_m \equiv 0$ near 0, and that each Δp_m is homogeneous of degree $m - 2$ for $m \geq 2$ (and is 0 for $m < 2$). From 1.30 we conclude $\Delta p_m \equiv 0$ for every m. We have thus represented u near 0 as an infinite sum of homogeneous harmonic polynomials.

Translating this local result from 0 to any other point in the domain of u, we have the following theorem.

1.31 Theorem: *Suppose u is harmonic on Ω and $a \in \Omega$. Then there exist harmonic homogeneous polynomials p_m of degree m such that*

1.32
$$u(x) = \sum_{m=0}^{\infty} p_m(x-a)$$

for all x near a, the series converging absolutely and uniformly near a.

Homogeneous expansions are better behaved than multiple power series. As we will see later (5.34), if u is harmonic on Ω and $B(a,r) \subset \Omega$, then the homogeneous expansion 1.32 is valid for all $x \in B(a,r)$. This is reminiscent of the standard power series result for holomorphic functions of one complex variable. Indeed, if u is holomorphic on $\Omega \subset \mathbf{R}^2 = \mathbf{C}$, then by the uniqueness of homogeneous expansions, 1.32 is precisely the holomorphic power series of u on $B(a,r)$.

Origin of the Term "Harmonic"

The word "harmonic" is commonly used to describe a quality of sound. Harmonic functions derive their name from a roundabout connection they have with one source of sound—a vibrating string.

Physicists label the movement of a point on a vibrating string "harmonic motion". Such motion may be described using sine and cosine functions, and in this context the sine and cosine functions are sometimes called *harmonics*. In classical Fourier analysis, functions on the unit circle are expanded in terms of sines and cosines. Analogous expansions exist on the sphere in $\mathbf{R}^n$, $n > 2$, in terms of homogeneous harmonic polynomials (see Chapter 5). Because these polynomials play the same role on the sphere that the harmonics sine and cosine play on the circle, they are called *spherical harmonics*. The term "spherical harmonic" was apparently first used in this context by William Thomson (Lord Kelvin) and Peter Tait (see [18], Appendix B). By the early 1900s, the word "harmonic" was applied not only to homogeneous polynomials with zero Laplacian, but to any solution of Laplace's equation.

Exercises

1. Show that if u and v are real-valued harmonic functions, then uv is harmonic if and only if $\nabla u \cdot \nabla v \equiv 0$.

2. Suppose Ω is connected and u is a real-valued harmonic function on Ω such that u^2 is harmonic. Prove that u is constant. Is this still true without the hypothesis that u is real valued?

3. Show that $\Delta(|x|^t) = t(t+n-2)|x|^{t-2}$.

4. *Laplacian in polar coordinates:* Suppose u is a twice continuously differentiable function of two real variables. Define a function U by $U(r,\theta) = u(r\cos\theta, r\sin\theta)$. Show that

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

5. *Laplacian in spherical coordinates:* Suppose u is a twice continuously differentiable function of three real variables. Define U by $U(\rho,\theta,\varphi) = u(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$. Show that

$$\Delta u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial U}{\partial \varphi}\right) + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2 U}{\partial \theta^2}.$$

6. Suppose g is a real-valued function in $C^2(\mathbf{R}^n)$ and $f \in C^2(\mathbf{R})$. Prove that

$$\Delta(f\circ g)(x) = f''(g(x))|\nabla g(x)|^2 + f'(g(x))\Delta g(x).$$

7. Show that if u is a positive function in $C^2(\Omega)$ and t is a constant, then

$$\Delta(u^t) = tu^{t-1}\Delta u + t(t-1)u^{t-2}|\nabla u|^2.$$

8. Show that if u, v are functions in $C^2(\Omega)$ with u positive, then

$$\begin{aligned}\Delta(u^v) = {}& vu^{v-1}\Delta u + u^v(\log u)\Delta v + v(v-1)u^{v-2}|\nabla u|^2\\ &+ u^v(\log u)^2|\nabla v|^2 + 2u^{v-1}(1+v\log u)\nabla u\cdot\nabla v.\end{aligned}$$

9. Suppose A is an m-by-n matrix of real numbers. Think of each $x \in \mathbf{R}^n$ as a column vector, so that Ax is then a column vector in $\mathbf{R}^m$. Show that

$$\Delta(|Ax|) = \frac{|Ax|^2|A|_2^2 - |A^t Ax|^2}{|Ax|^3},$$

where $|A|_2^2$ is the sum of the squares of the entries of A and A^t denotes the transpose of A.

10. Let u be harmonic on $\mathbf{R}^2$. Show that if f is holomorphic or conjugate holomorphic on $\mathbf{C}$, then $u \circ f$ is harmonic.

11. Suppose u is real valued and harmonic on B_2. For $(x, y) \in B_2$, define

$$v(x, y) = \int_0^y (D_1 u)(x, t)\, dt - \int_0^x (D_2 u)(t, 0)\, dt.$$

Show that $u + iv$ is holomorphic on B_2.

12. Suppose u is a harmonic function on Ω. Prove that the function $x \mapsto x \cdot \nabla u(x)$ is harmonic on Ω. (For a converse to this exercise, see Exercise 23 in Chapter 5.)

13. Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation such that $u \circ T$ is harmonic on $\mathbf{R}^n$ whenever u is harmonic on $\mathbf{R}^n$. Prove that T is a scalar multiple of an orthogonal transformation.

14. Suppose Ω is a bounded open subset of $\mathbf{R}^n$ with smooth boundary and u is a smooth function on $\overline{\Omega}$ such that $\Delta(\Delta u) = 0$ on Ω and $u = D_{\mathbf{n}} u = 0$ on $\partial\Omega$. Prove that $u = 0$.

15. Suppose that Ω is connected and that u is real valued and harmonic on Ω. Show that if u is nonconstant on Ω, then $u(\Omega)$ is open in $\mathbf{R}$. (Thus u is an open mapping from Ω to $\mathbf{R}$.)

16. Suppose Ω is bounded and $\partial\Omega$ is connected. Show that if u is a real-valued continuous function on $\overline{\Omega}$ that is harmonic on Ω, then $u(\Omega) \subset u(\partial\Omega)$. Is this true for complex-valued u?

17. A function is called *radial* if its value at x depends only on $|x|$. Prove that a radial harmonic function on B is constant.

18. Give another proof that $\int_S P(x,\zeta)\,d\sigma(\zeta) = 1$ for every $x \in B$ by showing that the function $x \mapsto \int_S P(x,\zeta)\,d\sigma(\zeta)$ is harmonic and radial on B.

19. Show that $P[f \circ T] = P[f] \circ T$ for every $f \in C(S)$ and every orthogonal transformation T.

20.. Find the Poisson kernel for the ball $B(a,R)$.

21. Use the mean-value property and its converse to give another proof that the uniform limit of a sequence of harmonic functions is harmonic.

22. Suppose u is a continuous function on Ω, and that for each $x \in \Omega$ there is a sequence of positive numbers $r_j \to 0$ such that

$$u(x) = \frac{1}{V(B(x,r_j))}\int_{B(x,r_j)} u\,dV$$

for each j. Prove that u is harmonic on Ω.

23. *One-Radius Theorem:* Suppose u is continuous on $\overline{B}$ and that for every $x \in B$, there exists a positive number $r(x) \leq 1 - |x|$ such that

$$u(x) = \int_S u(x + r(x)\zeta)\,d\sigma(\zeta).$$

Prove that u is harmonic on B.

24. Show that the one-radius theorem fails if the assumption "u is continuous on $\overline{B}$" is relaxed to "u is continuous on B". (*Hint suggested by Walter Rudin:* When $n = 2$, set $u(x) = a_m + b_m \log|x|$ on the annulus $\{1 - 2^{-m} \leq |x| \leq 1 - 2^{-m-1}\}$, where the constants a_m, b_m are chosen inductively. Proceed analogously when $n > 2$.)

25. *Hopf Lemma:* Suppose that u is real valued, nonconstant, and harmonic on $\overline{B}$. Show that if u attains its maximum value on $\overline{B}$ at $\zeta \in S$, then there is a positive constant c such that

$$u(\zeta) - u(r\zeta) \geq c(1 - r)$$

for all $r \in (0,1)$. Conclude that $(D_{\mathbf{n}}u)(\zeta) > 0$.

26. Show that the previous exercise can fail if instead of having a maximum at ζ, the restriction $u|_S$ has only a strict local maximum at ζ. (*Hint:* Take $n = 2$ and $u(x, y) = x^2 - y^2 - 3x$.)

27. Prove that a harmonic function on $\overline{B}$ whose normal derivative vanishes identically on S is constant.

28. Show that the previous result holds if the ball is replaced by a bounded smooth domain in $\mathbf{R}^n$ that has an internally tangent ball at each boundary point.

29. Show that a polynomial p is homogeneous of degree m if and only if $x \cdot \nabla p = mp$.

30. Prove that if p is a harmonic polynomial on $\mathbf{R}^n$ that is homogeneous of degree m, then $p/|x|^{2m+n-2}$ is harmonic on $\mathbf{R}^n \setminus \{0\}$.

31. Suppose that $\sum c_\alpha x^\alpha$ converges in $R(y)$. Prove that $\sum c_\alpha x^\alpha$ is real analytic in $R(y)$.

32. A function $u\colon \Omega \to \mathbf{R}^m$ is called real analytic if each component of u is real analytic. Prove that the composition of real-analytic functions is real analytic. Deduce, as a corollary, that sums, products, and quotients of real-analytic functions are real analytic.

33. Let m be a positive integer. Characterize all real-analytic functions u on $\mathbf{R}^n$ such that $u(tx) = t^m u(x)$ for all $x \in \mathbf{R}^n$ and all $t \in \mathbf{R}$.

34. Show that the power series expansion of a function harmonic on $\mathbf{R}^n$ converges everywhere on $\mathbf{R}^n$.

CHAPTER 2

Bounded Harmonic Functions

Liouville's Theorem

Liouville's Theorem in complex analysis states that a bounded holomorphic function on **C** is constant. A similar result holds for harmonic functions on $\mathbf{R}^n$. The simple proof given below is taken from Edward Nelson's paper [13], which is one of the rare mathematics papers not containing a single mathematical symbol.

2.1 Liouville's Theorem: *A bounded harmonic function on* $\mathbf{R}^n$ *is constant.*

PROOF: Suppose u is a harmonic function on $\mathbf{R}^n$, bounded by M. Let $x \in \mathbf{R}^n$ and let $r > 0$. By the volume version of the mean-value property (Theorem 1.6),

$$
\begin{aligned}
|u(x) - u(0)| &= \frac{1}{V(B(0,r))} \left| \int_{B(x,r)} u\, dV - \int_{B(0,r)} u\, dV \right| \\
&\leq M \frac{V(\mathcal{D}_r)}{V(B(0,r))},
\end{aligned}
$$

where $\mathcal{D}_r$ denotes the symmetric difference of $B(x,r)$ and $B(0,r)$, so that $\mathcal{D}_r = [B(x,r) \cup B(0,r)] \setminus [B(x,r) \cap B(0,r)]$. The last expression above tends to 0 as $r \to \infty$. Thus $u(x) = u(0)$, and so u is constant. ■

Liouville's Theorem leads to an easy proof of a uniqueness theorem for bounded harmonic functions on open half-spaces. The upper half-

space $H = H_n$ is the open subset of $\mathbf{R}^n$ defined by

$$H = \{x \in \mathbf{R}^n : x_n > 0\}.$$

In this setting we often identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$, writing a typical point $z \in \mathbf{R}^n$ as $z = (x, y)$, where $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$. We also identify ∂H with $\mathbf{R}^{n-1}$.

A harmonic function on a compact set is determined by its restriction to the boundary (this follows from the maximum principle). However, a harmonic function on a closed half-space is not determined by restriction to the boundary. For example, the harmonic function u on $\overline{H}$ defined by $u(x, y) = y$ agrees on the boundary of the half-space with the harmonic function 0. The next result shows that this behavior cannot occur if we consider only harmonic functions that are bounded.

2.2 Corollary: *Suppose u is a continuous bounded function on $\overline{H}$ that is harmonic on H. If $u = 0$ on ∂H, then $u \equiv 0$ on $\overline{H}$.*

PROOF: For $x \in \mathbf{R}^{n-1}$ and $y < 0$, define $u(x, y) = -u(x, -y)$, thereby extending u to a bounded continuous function defined on all of $\mathbf{R}^n$. Clearly u satisfies the local mean-value property specified in Theorem 1.24, so u is harmonic on $\mathbf{R}^n$. Liouville's Theorem (2.1) now shows that u is constant on $\mathbf{R}^n$. ∎

In Chapter 7 we will study harmonic functions on half-spaces in detail.

Isolated Singularities

Everyone knows that an isolated singularity of a bounded holomorphic function is removable. We now show that the same is true for bounded harmonic functions.

We call a point $a \in \Omega$ an *isolated singularity* of any function u defined on $\Omega \setminus \{a\}$. When u is harmonic on $\Omega \setminus \{a\}$, the isolated singularity a is said to be *removable* if u has a harmonic extension to Ω.

2.3 Theorem: *An isolated singularity of a bounded harmonic function is removable.*

PROOF: It suffices to show that if u is bounded and harmonic on $\overline{B}\setminus\{0\}$, then u has a harmonic extension to B. Without loss of generality, we can assume that u is real valued. The only candidate for a harmonic extension of u to B is the Poisson integral $P[u|_S]$.

Assume first that $n > 2$. For $\varepsilon > 0$, define the harmonic function v_ε on $B \setminus \{0\}$ by

$$v_\varepsilon(x) = u(x) - P[u|_S](x) + \varepsilon(|x|^{2-n} - 1).$$

Observe that as $|x| \to 1$, we have $v_\varepsilon(x) \to 0$ (by 1.17), while the boundedness of u shows that $v_\varepsilon(x) \to \infty$ as $x \to 0$. By Corollary 1.10 (with "lim sup" replaced by "lim inf"), $v_\varepsilon \geq 0$ in $B \setminus \{0\}$. Letting $\varepsilon \to 0$, we conclude that $u - P[u|_S] \geq 0$ on $B \setminus \{0\}$. Replacing u by $-u$, we also have $u - P[u|_S] \leq 0$, giving $u = P[u|_S]$ on $B \setminus \{0\}$. Thus $P[u|_S]$ is the desired harmonic extension of u to B.

The proof when $n = 2$ is the same, except that $(|x|^{2-n} - 1)$ should be replaced by $\log 1/|x|$. ■

Cauchy's Estimates

If f is holomorphic and bounded by M on a disk $B(a, r) \subset \mathbf{C}$, then

$$|f^{(m)}(a)| \leq \frac{m!\,M}{r^m}$$

for every nonnegative integer m; these are Cauchy's Estimates from complex analysis. The next theorem gives the comparable results for harmonic functions defined on balls in $\mathbf{R}^n$.

2.4 Cauchy's Estimates: *Let α be a multi-index. Then there is a positive constant C_α such that*

$$|D^\alpha u(a)| \leq \frac{C_\alpha M}{r^{|\alpha|}}$$

for all functions u harmonic and bounded by M on $B(a, r)$.

PROOF: We can assume that $a = 0$. If u is harmonic and bounded by M on $\overline{B}$, then by 1.22 we have

$$\begin{aligned}|D^\alpha u(0)| &= \left|\int_S u(\zeta) D^\alpha P(0,\zeta)\, d\sigma(\zeta)\right| \\ &\le M \int_S |D^\alpha P(0,\zeta)|\, d\sigma(\zeta) \\ &= C_\alpha M,\end{aligned}$$

where $C_\alpha = \int_S |D^\alpha P(0,\zeta)|\, d\sigma(\zeta)$.

If u is harmonic and bounded by M on $\overline{B}(0,r)$, then applying the result in the previous paragraph to the r-dilate u_r shows that

$$|D^\alpha u(0)| \le \frac{C_\alpha M}{r^{|\alpha|}}.$$

Replacing r by $r - \varepsilon$ and letting ε decrease to 0, we obtain the same conclusion if u is harmonic on the open ball $B(0,r)$ and bounded by M there. ∎



Augustin-Louis Cauchy (1789–1857), whose collected works consisting of 789 mathematics papers fill 27 volumes, made major contributions to the study of harmonic functions.

The following corollary shows that the derivatives of a bounded harmonic function on Ω cannot grow too fast near $\partial\Omega$. We let $d(a, E)$ denote the distance from a point a to a set E.

2.5 Corollary: *Let u be a bounded harmonic function on Ω, and let α be a multi-index. Then there is a constant C such that*

$$|D^{\alpha}u(a)| \leq \frac{C}{d(a,\partial\Omega)^{|\alpha|}}$$

for all $a \in \Omega$.

PROOF: For each $a \in \Omega$, apply Cauchy's Estimates (Theorem 2.4) with $r = d(a,\partial\Omega)$. ∎

Normal Families

In complex analysis the term *normal family* refers to a collection of holomorphic functions with the property that every sequence in the collection contains a subsequence converging uniformly on compact subsets of the domain. The most useful result in this area (and the key tool in most proofs of the Riemann Mapping Theorem) states that a collection of holomorphic functions that is uniformly bounded on each compact subset of the domain is a normal family. We now prove the analogous result for harmonic functions.

2.6 Theorem: *If (u_m) is a sequence of harmonic functions on Ω that is uniformly bounded on each compact subset of Ω, then some subsequence of (u_m) converges uniformly on each compact subset of Ω.*

PROOF: The key to the proof is the following observation: there exists a constant $C < \infty$ such that for all u harmonic and bounded by M on any ball $B(a, 2r)$,

$$|u(x) - u(a)| \leq \big(\sup_{B(a,r)} |\nabla u|\big)|x - a| \leq \frac{CM}{r}|x - a|$$

for all $x \in B(a,r)$. The first inequality is standard from advanced calculus; the second inequality follows from Cauchy's Estimates (2.4).

Now suppose $K \subset \Omega$ is compact, and let $r = d(K,\partial\Omega)/3$. Because the set $K_{2r} = \{x \in \mathbf{R}^n : d(x,K) \leq 2r\}$ is a compact subset of Ω, the sequence (u_m) is uniformly bounded by some $M < \infty$ on K_{2r}. Let $a, x \in K$ and assume $|x - a| < r$. Then $x \in B(a,r)$ and $|u_m| \leq M$ on

$B(a, 2r) \subset K_{2r}$ for all m, and so we conclude from the first paragraph that

$$|u_m(x) - u_m(a)| \leq \frac{CM}{r}|x - a|$$

for all m. It follows that the sequence (u_m) is equicontinuous on K.

To finish, choose compact sets

$$K_1 \subset K_2 \subset \cdots \subset \Omega$$

whose interiors cover Ω. Because (u_m) is equicontinuous on K_1, the Arzela-Ascoli Theorem ([15], Theorem 11.28) implies (u_m) contains a subsequence that converges uniformly on K_1. Applying Arzela-Ascoli again, there is a subsequence of this subsequence converging uniformly on K_2, and so on. If we list these subsequences one after another in rows, then the subsequence obtained by traveling down the diagonal converges uniformly on each K_j, and hence on each compact subset of Ω. ■

Note that by Theorem 1.23, the convergent subsequence obtained above converges to a harmonic function; furthermore, every partial derivative of this subsequence converges uniformly on each compact subset of Ω.

Theorem 2.6 is often useful in showing that certain extrema exist. For example, if $a \in \Omega$, then there exists a harmonic function v on Ω such that $|v| < 1$ on Ω and

$$|\nabla v(a)| = \sup\{|\nabla u(a)| : u \text{ is harmonic on } \Omega \text{ and } |u| < 1 \text{ on } \Omega\}.$$

Maximum Principles

Corollary 1.10 is the maximum principle in its most general form. It states that if u is a real-valued harmonic function on Ω and $u \leq M$ at the "boundary" of Ω, then $u \leq M$ on Ω. The catch is that we need to consider ∞ as a boundary point. (Again, the function $u(x, y) = y$ on H shows why this is necessary.) Often it is possible to ignore the point at infinity when u is bounded; the next result shows that this is always possible in two dimensions.

2.7 Theorem: *Suppose $\Omega \subset \mathbf{R}^2$ and $\Omega \neq \mathbf{R}^2$. If u is a real-valued, bounded harmonic function on Ω satisfying*

2.8
$$\limsup_{k\to\infty} u(a_k) \le M$$

for every sequence (a_k) in Ω converging to a point in $\partial\Omega$, then $u \le M$ on Ω.

PROOF: Because $\Omega \neq \mathbf{R}^2$, $\partial\Omega$ is not empty. Let $\varepsilon > 0$, and choose a sequence in Ω converging to a point in $\partial\Omega$. By hypothesis, u is less than $M + \varepsilon$ on the tail end of this sequence. It follows that there is a closed ball $\overline{B}(a, r) \subset \Omega$ on which $u < M + \varepsilon$.

Define $\Omega' = \Omega \setminus \overline{B}(a, r)$, and set

$$v(z) = \log\left|\frac{z-a}{r}\right|$$

for $z \in \Omega'$. Then v is positive and harmonic on Ω', and $v(z) \to \infty$ as $z \to \infty$ within Ω'.

For $t > 0$, we now define the harmonic function w_t on Ω' by

$$w_t = u - M - \varepsilon - tv.$$

By 2.8 and the preceding remarks, $\limsup_{k\to\infty} w_t(a_k) < 0$ for every sequence (a_k) in Ω' converging to a point in $\partial\Omega'$, while the boundedness of u on Ω' shows that $w_t(a_k) \to -\infty$ for every sequence (a_k) converging to ∞ within Ω'. By Corollary 1.10, $w_t < 0$ on Ω'.

We now let $t \to 0$ to obtain $u \le M + \varepsilon$ on Ω'. Because $u < M + \varepsilon$ on $\overline{B}(a, r)$, we have $u \le M + \varepsilon$ on all of Ω. Finally, since ε is arbitrary, $u \le M$ on Ω, as desired. ∎

The higher-dimensional analogue of Theorem 2.7 fails. For an example, let $\Omega = \{x \in \mathbf{R}^n : |x| > 1\}$ and set $u(x) = 1 - |x|^{2-n}$. If $n > 2$, then u is a bounded harmonic function on $\overline{\Omega}$ that vanishes on $\partial\Omega$ but is not identically 0 on Ω. (In fact, u is never zero on Ω.)

The proof of Theorem 2.7 carries over to higher dimensions except for one key point. Specifically, when $n = 2$, there exists a positive harmonic function v on $\mathbf{R}^n \setminus \overline{B}$ such that $v(z) \to \infty$ as $z \to \infty$. When $n > 2$, there exists no such v; in fact, every positive harmonic function on $\mathbf{R}^n \setminus \overline{B}$ has a finite limit at ∞ when $n > 2$ (Theorem 4.10).

The following maximum principle is nevertheless valid in all dimensions. Recall that H_n denotes the upper half-space of $\mathbf{R}^n$.

2.9 Theorem: *Suppose $\Omega \subset H_n$. If u is a real-valued, bounded harmonic function on Ω satisfying*

$$\limsup_{k\to\infty} u(a_k) \le M$$

for every sequence (a_k) in Ω converging to a point in $\partial\Omega$, then $u \le M$ on Ω.

PROOF: For $(x, y) \in \Omega$, define

$$v(x,y) = \sum_{k=1}^{n-1} \log(x_k{}^2 + (y+1)^2).$$

Then v is positive and harmonic on Ω, and $v(z) \to \infty$ as $z \to \infty$ within H_n. Having obtained v, we can use the ideas in the proof of Theorem 2.7 to finish the proof. The details are even easier here and we leave them to the reader. ∎

Limits Along Rays

We now apply some of the preceding results to study the boundary behavior of harmonic functions defined in the upper half-plane H_2. We will need the notion of a nontangential limit, which for later purposes we define for functions on half-spaces of arbitrary dimension.

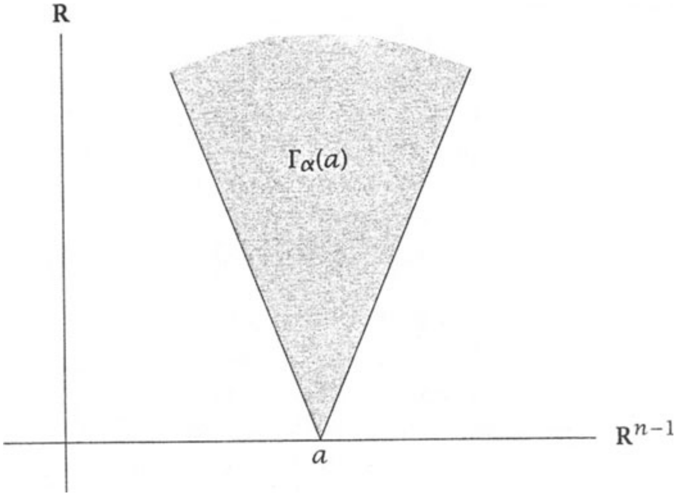
Given $a \in \mathbf{R}^{n-1}$ and $\alpha > 0$, set

$$\Gamma_\alpha(a) = \{(x,y) \in H_n : |x-a| < \alpha y\}.$$

Geometrically, $\Gamma_\alpha(a)$ is a cone with vertex a and axis of symmetry parallel to the y-axis.

We have $\Gamma_\alpha(a) \subset \Gamma_\beta(a)$ if $\alpha < \beta$, and H_n is the union of the sets $\Gamma_\alpha(a)$ as α ranges over $(0, \infty)$.

A function u defined on H_n is said to have a *nontangential limit* L at $a \in \mathbf{R}^{n-1}$ if for every $\alpha > 0$, $u(z) \to L$ as $z \to a$ within $\Gamma_\alpha(a)$. The term "nontangential" is used because no curve in $\Gamma_\alpha(a)$ that approaches a



The cone $\Gamma_\alpha(a)$.

can be tangent to ∂H_n. Exercise 17 of this chapter shows that a bounded harmonic function on H_n can have a nontangential limit at a point of ∂H_n even though the ordinary limit does not exist at that point.

The following theorem for bounded harmonic functions on H_2 asserts that a nontangential limit can be deduced from a limit along a certain one-dimensional set.

2.10 Theorem: *Suppose that u is bounded and harmonic on H_2. If $0 < \theta_1 < \theta_2 < \pi$ and*

$$\lim_{r \to 0} u(re^{i\theta_1}) = L = \lim_{r \to 0} u(re^{i\theta_2}),$$

then u has nontangential limit L at 0.

PROOF: We may assume $L = 0$.

If the theorem is false, then for some $\alpha > 0$, $u(z)$ fails to have limit 0 as $z \to 0$ within $\Gamma_\alpha(0)$. This means that there exists an $\varepsilon > 0$ and a sequence (z_j) tending to 0 within $\Gamma_\alpha(0)$ such that

2.11 $$|u(z_j)| > \varepsilon$$

for all j.

Define $K = [-\alpha, \alpha] \times \{1\}$, and write $z_j = r_j w_j$, where $w_j \in K$ and $r_j > 0$. Because $z_j \to 0$, we have $r_j \to 0$.

Setting $u_j(z) = u(r_j z)$, note that (u_j) is a uniformly bounded sequence of harmonic functions on H_2. By Theorem 2.6, there exists a subsequence of (u_j) that converges uniformly on compact subsets of H_2 to a bounded harmonic function v on H_2; for simplicity we denote this subsequence by (u_j) as well.

Examining the limit function v, we see that

$$v(re^{i\theta_1}) = \lim_{j \to \infty} u_j(re^{i\theta_1}) = \lim_{j \to \infty} u(r_j re^{i\theta_1}) = 0$$

for all $r > 0$. Similarly, $v(re^{i\theta_2}) = 0$ for all $r > 0$. The reader may now be tempted to apply Theorem 2.7 to the region between the two rays; unfortunately we do not know that $v(z) \to 0$ as $z \to 0$ between the given rays. We avoid this problem by observing that the function $z \mapsto v(e^z)$ is bounded and harmonic on the strip $\Omega = \{z = x + iy : \theta_1 < y < \theta_2\}$, and that $v(e^z)$ extends continuously to $\overline{\Omega}$ with $v(e^z) = 0$ on $\partial\Omega$. By Theorem 2.7, $v(e^z) \equiv 0$ on Ω, and thus $v \equiv 0$ on H_2.

The sequence (u_j) therefore converges to 0 uniformly on compact subsets of H_2. In particular, $u_j \to 0$ uniformly on K. It follows that $u_j(w_j) = u(z_j) \to 0$ as $j \to \infty$, contradicting 2.11. ∎

Does a limit along one ray suffice to give a nontangential limit in Theorem 2.10? To see that the answer is no, consider the bounded harmonic function u on H_2 defined by $u(re^{i\theta}) = \theta$ for $\theta \in (0, \pi)$. This function has a limit along each ray in H_2 emanating from the origin, but different rays yield different limits. (One ray will suffice for a bounded holomorphic function; see Exercise 22 of this chapter.)

Bounded Harmonic Functions on the Ball

In the last chapter we defined the Poisson integral $P[f]$ assuming that f is continuous on S. We can easily enlarge the class of functions f for which $P[f]$ is defined. For example, if f is a bounded (Borel) measurable function on S, then

$$P[f](x) = \int_S f(\zeta) P(x, \zeta)\, d\sigma(\zeta)$$

defines a bounded harmonic function on B; we leave the verification to the reader.

Allowing bounded measurable boundary data gives us many more examples of bounded harmonic functions on B than could otherwise be obtained. For example, in Chapter 6 we will see that the extremal function in the Schwarz Lemma for harmonic functions is the Poisson integral of a bounded discontinuous function on S. In that chapter we will also prove a fundamental result (Theorem 6.13): given a bounded harmonic function u on B, there exists a bounded measurable f on S such that $u = P[f]$ on B.

Exercises

1. Give an example of a bounded harmonic function on B that is not uniformly continuous on B.

2. (a) Suppose u is a harmonic function on $B \setminus \{0\}$ such that

$$|x|^{n-2}u(x) \to 0 \quad \text{as } x \to 0.$$

Prove that u has a removable singularity at 0.

(b) Suppose u is a harmonic function on $B_2 \setminus \{0\}$ such that

$$u(x)/\log|x| \to 0 \quad \text{as } x \to 0.$$

Prove that u has a removable singularity at 0.

3. Suppose that u is harmonic on $\mathbf{R}^n$ and that $u(x,0) = 0$ for all $x \in \mathbf{R}^{n-1}$. Prove that $u(x,-y) = -u(x,y)$ for all $(x,y) \in \mathbf{R}^n$.

4. Under what circumstances can a function harmonic on $\mathbf{R}^n$ vanish on the union of two hyperplanes?

5. Use Cauchy's Estimates (Theorem 2.4) to give another proof of Liouville's Theorem (Theorem 2.1).

6. Let K be a compact subset of Ω and let α be a multi-index. Show that there is a constant $C = C(\Omega, K, \alpha)$ such that

$$|D^\alpha u(a)| \le C \sup\{|u(x)| : x \in \Omega\}$$

for every function u harmonic on Ω and every $a \in K$.

7. Suppose u is harmonic on $\mathbf{R}^n$ and $|u(x)| \le A(1 + |x|^p)$ for all $x \in \mathbf{R}^n$, where A is a constant and $p \ge 0$. Prove that u is a polynomial of degree at most p.

8. Prove if (u_m) is a pointwise convergent sequence of harmonic functions on Ω that is uniformly bounded on each compact subset of Ω, then (u_m) converges uniformly on each compact subset of Ω.

9. Show that if u is the pointwise limit of a sequence of harmonic functions on Ω, then u is harmonic on a dense open subset of Ω. (Hint: Baire's Theorem.)

10. Let u be a bounded harmonic function on B. Prove that

$$\sup_{x \in B}(1 - |x|)|\nabla u(x)| < \infty.$$

11. The set of harmonic functions u on B satisfying the inequality in Exercise 10 is called the *harmonic Bloch space*. Prove the harmonic Bloch space is a Banach space under the norm defined by

$$\|u\| = \sup_{x \in B}(1 - |x|)|\nabla u(x)| + |u(0)|.$$

12. Give an example of an unbounded harmonic function in the harmonic Bloch space.

13. Prove that if u is in the harmonic Bloch space and α is a multi-index with $|\alpha| > 0$, then

$$\sup_{x \in B}(1 - |x|)^{|\alpha|}|D^\alpha u(x)| < \infty.$$

14. For $a \in B$, let B_a denote the ball centered a with radius $\frac{1-|a|}{2}$. Prove that if u is harmonic on B, then u is in the harmonic Bloch space if and only if

$$\sup_{a \in B} \frac{1}{V(B_a)} \int_{B_a} |u - u(a)|\, dV < \infty.$$

15. Let $\mathcal{U}$ denote the set of harmonic functions u on B such that $u(0) = 0$ and

$$\sup_{x \in B}(1 - |x|)|\nabla u(x)| \le 1.$$

Prove that there exists a function $v \in \mathcal{U}$ such that

$$\int_S |v(\zeta/2)|\, d\sigma(\zeta) = \sup_{u \in \mathcal{U}} \int_S |u(\zeta/2)|\, d\sigma(\zeta).$$

16. Suppose $\Omega \subset H_n$ and that u is a continuous bounded function on $\overline{\Omega}$ that is harmonic on Ω. Prove that if $u = 0$ on $\partial\Omega$, then $u \equiv 0$ on $\overline{\Omega}$.

17. Let $f(z) = e^{-i/z}$. Show that f is a bounded holomorphic function on H_2, that f has a nontangential limit at the origin, but that f does not have a limit along some curve in H_2 terminating at the origin.

18. Suppose $0 < \theta_1 < \theta_2 < \pi$ and $L_1, L_2 \in \mathbf{C}$. Show that there is a bounded harmonic function u on H_2 such that $u(re^{i\theta_k}) \to L_k$ as $r \to 0$ for $k = 1, 2$.

19. Suppose u is a bounded harmonic function on H_2 with limits at 0 along two distinct rays. Specifically, suppose

$$\lim_{r \to 0} u(re^{i\theta_k}) = L_k$$

for $k = 1, 2$, where $0 < \theta_1 < \theta_2 < \pi$. Show that $\lim_{r \to 0} u(re^{i\theta})$ exists for every $\theta \in (0, \pi)$, and evaluate this limit as a function of θ.

20. Define $f(z) = e^{i \log z}$, where $\log z$ denotes the principal-valued logarithm. Show that f is a bounded holomorphic function on H_2 whose real and imaginary parts fail to have a limit along every ray in H_2 emanating from the origin.

21. Let $\theta_0 \in (0, \pi)$. Prove that there exists a bounded harmonic function u on H_2 such that $\lim_{r \to 0} u(re^{i\theta})$ exists if and only if $\theta = \theta_0$. (*Hint:* Do this first for $\theta_0 = \pi/2$ by letting $u(z) = \operatorname{Re} e^{i \log z}$ and considering $u(x, y) - u(-x, y)$.)

22. Let f be a bounded holomorphic function on H_2, and suppose $\lim_{r \to 0} f(re^{i\theta})$ exists for some $\theta \in (0, \pi)$. Prove that f has a nontangential limit at 0.

23. Let u be a bounded harmonic function on H_3 that has the same limit along two distinct rays in H_3 emanating from 0. Need u have a nontangential limit at 0?

24. Prove that when $n > 2$ there does not exist a harmonic function v on $\mathbf{R}^n \setminus \overline{B}$ such that $v(z) \to \infty$ as $z \to \infty$.

25. Let K denote a compact line segment contained in B_3. Show that every bounded harmonic function on $B_3 \setminus K$ extends to be harmonic on B_3.

CHAPTER 3

Positive Harmonic Functions

This chapter focuses on the special properties of positive harmonic functions. We will describe the positive harmonic functions defined on all of $\mathbf{R}^n$ (Liouville's Theorem), show that positive harmonic functions cannot oscillate wildly (Harnack's Inequality), and characterize the behavior of positive harmonic functions near isolated singularities (Bôcher's Theorem).

Liouville's Theorem

In Chapter 2 we proved that a bounded harmonic function on $\mathbf{R}^n$ is constant. We now improve that result. In Chapter 9 we will improve even the result below (see 9.10).

3.1 Liouville's Theorem for Positive Harmonic Functions: *A positive harmonic function on* $\mathbf{R}^n$ *is constant.*

PROOF: The proof is a bit more delicate than that given for bounded harmonic functions. Let u be a positive harmonic function defined on $\mathbf{R}^n$. Fix $x \in \mathbf{R}^n$. Let $r > |x|$, and let $\mathcal{D}_r$ denote the symmetric difference of the balls $B(x,r)$ and $B(0,r)$. The volume version of the mean-value property (1.6) shows that

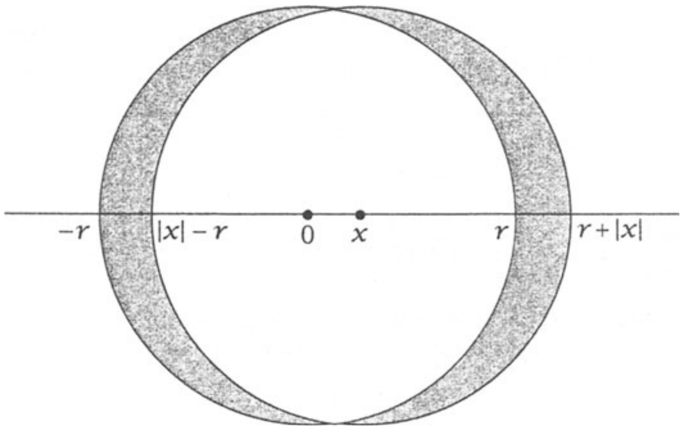
$$u(x) - u(0) = \frac{1}{V(B(0,r))}\Big[\int_{B(x,r)} u\,dV - \int_{B(0,r)} u\,dV\Big].$$

Because the integrals of u over $B(x,r) \cap B(0,r)$ cancel (see 3.2), we have

$$\begin{aligned}
|u(x) - u(0)| &\leq \frac{1}{V(B(0,r))} \int_{\mathcal{D}_r} u\,dV \\
&\leq \frac{1}{V(B(0,r))} \int_{B(0,r+|x|)\setminus B(0,r-|x|)} u\,dV \\
&= \frac{1}{V(B(0,r))} \Big[\int_{B(0,r+|x|)} u\,dV - \int_{B(0,r-|x|)} u\,dV \Big] \\
&= u(0) \frac{(r+|x|)^n - (r-|x|)^n}{r^n}.
\end{aligned}$$

Note that the positivity of u was used in the first inequality.

Now letting $r \to \infty$, we see that $u(x) = u(0)$, proving that u is constant. ∎



3.2 *The symmetric difference $\mathcal{D}_r$ (shaded) of $B(x,r)$ and $B(0,r)$.*

Liouville's Theorem for positive harmonic functions leads to an easy proof that a positive harmonic function on $\mathbf{R}^2 \setminus \{0\}$ is constant.

3.3 Corollary: *A positive harmonic function on $\mathbf{R}^2 \setminus \{0\}$ is constant.*

PROOF: If u is positive and harmonic on $\mathbf{R}^2 \setminus \{0\}$, then the function $z \mapsto u(e^z)$ is positive and harmonic on $\mathbf{R}^2$ (= $\mathbf{C}$) and hence (by 3.1) is constant. This proves that u is constant. ∎

The higher-dimensional analogue of Corollary 3.3 fails; for example, the function $|x|^{2-n}$ is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$ when $n > 2$. We will classify the positive harmonic functions on $\mathbf{R}^n \setminus \{0\}$ for $n > 2$ after the proof of Bôcher's Theorem; see Corollary 3.14.

Harnack's Inequality and Harnack's Principle

Positive harmonic functions cannot oscillate too much on a compact set $K \subset \Omega$ if Ω is connected; the precise statement is Harnack's Inequality (3.6). We first consider the important special case where Ω is the open unit ball.

3.4 Harnack's Inequality for the Ball: *If u is positive and harmonic on B, then*

$$\frac{1-|x|}{(1+|x|)^{n-1}}u(0) \le u(x) \le \frac{1+|x|}{(1-|x|)^{n-1}}u(0)$$

for all $x \in B$.

PROOF: If u is positive and harmonic on the closed unit ball $\overline{B}$, then

$$\begin{aligned} u(x) &= P[u|_S](x) \\ &= \int_S u(\zeta)\frac{1-|x|^2}{|x-\zeta|^n}\,d\sigma(\zeta) \\ &\le \frac{1-|x|^2}{(1-|x|)^n}\int_S u(\zeta)\,d\sigma(\zeta) \\ &= \frac{1+|x|}{(1-|x|)^{n-1}}u(0) \end{aligned}$$

for all $x \in B$. If u is positive and harmonic on B, apply the estimate above to the dilates u_r and take the limit as $r \to 1$. This gives us the second inequality of the theorem. The first inequality is proved similarly. ■

Define $\alpha(t) = (1-t)/(1+t)^{n-1}$ and $\beta(t) = (1+t)/(1-t)^{n-1}$. After a translation and a dilation, 3.4 tells us that if u is positive and harmonic on $B(a, R)$, and $|x - a| \le r < R$, then

3.5 $$\alpha(r/R)u(a) \le u(x) \le \beta(r/R)u(a).$$

3.6 Harnack's Inequality: *Suppose that Ω is connected and that K is a compact subset of Ω. Then there is a constant $C \in (1, \infty)$ such that*

$$\frac{1}{C} \le \frac{u(y)}{u(x)} \le C$$

for all points x and y in K and all positive harmonic functions u on Ω.

PROOF: We will prove that there is a constant $C \in (1, \infty)$ such that $u(y)/u(x) \le C$ for all $x, y \in K$ and all positive harmonic functions u on Ω. Because x and y play symmetric roles, the other inequality will also hold.

For $(x, y) \in \Omega \times \Omega$, define

$$s(x, y) = \sup\{u(y)/u(x) : u \text{ is positive and harmonic on } \Omega\}.$$

We first show that $s < \infty$ on $\Omega \times \Omega$.

Fix $x \in \Omega$, and define

$$E = \{y \in \Omega : s(x, y) < \infty\}.$$

Because $x \in E$, E is not empty. If $y \in E$, we may choose $r > 0$ such that $B(y, 2r) \subset \Omega$. By 3.5, $u \le \beta(1/2)u(y)$ on $B(y, r)$ for all positive harmonic functions u on Ω. We then have $B(y, r) \subset E$, proving that E is open. If $z \in \Omega$ is a limit point of E, there exists an $r > 0$ and a $y \in E$ such that $z \in B(y, r) \subset B(y, 2r) \subset \Omega$. By 3.5, $u(z) \le \beta(1/2)u(y)$ for all positive harmonic functions u on Ω. We then have $z \in E$, proving that E is closed. The connectivity of Ω therefore shows that $E = \Omega$.

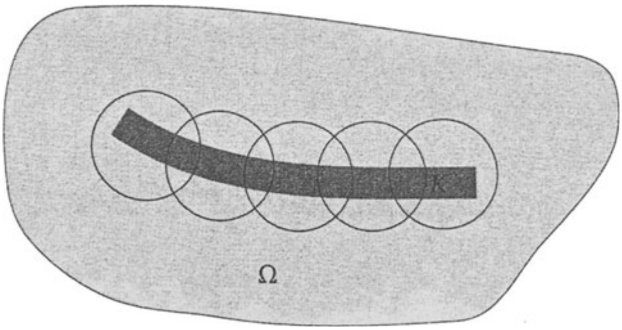
We now know that s is finite at every point of $\Omega \times \Omega$. Let $K \subset \Omega$ be compact, and let $(a, b) \in K \times K$. Then by 3.5,

$$\frac{u(y)}{u(x)} \le \frac{\beta(1/2)u(b)}{\alpha(1/2)u(a)} \le \frac{\beta(1/2)}{\alpha(1/2)}s(a, b)$$

for all (x, y) in a neighborhood of (a, b), and for all positive harmonic functions u on Ω. Because $K \times K$ is covered by finitely many such neighborhoods, s is bounded above on $K \times K$, as desired. ∎

Note that the constant C in 3.6 may depend upon Ω and K, but that C is independent of x, y, and u.

An intuitive way to remember Harnack's Inequality is shown in 3.7. Here we have covered K with a finite chain of overlapping balls (possible, since Ω is connected); to compare the values of a positive harmonic function at any two points in K, we can think of a finite chain of inequalities of the kind expressed in 3.5.



3.7 *K covered by overlapping balls.*

Harnack's Inequality leads to an important convergence theorem for harmonic functions known as Harnack's Principle. Consider a monotone sequence of continuous functions on Ω. The pointwise limit of such a sequence need not behave well—it could be infinite at some points and finite at other points. Even if it is finite everywhere, there is no reason to expect that our sequence converges uniformly on every compact subset of Ω. Harnack's Principle shows that none of this bad behavior can occur for a monotone sequence of harmonic functions.

3.8 Harnack's Principle: *Suppose Ω is connected and (u_m) is a pointwise increasing sequence of harmonic functions on Ω. Then either (u_m) converges uniformly on compact subsets of Ω to a function harmonic on Ω or $u_m(x) \to \infty$ for every $x \in \Omega$.*

PROOF: Replacing u_m by $u_m - u_1 + 1$, we can assume that each u_m is positive on Ω. Set $u(x) = \lim_{m\to\infty} u_m(x)$ for each $x \in \Omega$.

First suppose u is finite everywhere on Ω. Let K be a compact subset of Ω. Fix $x \in K$. Harnack's Inequality (3.6) shows there is a constant $C \in (1, \infty)$ such that

$$u_m(y) - u_k(y) \leq C(u_m(x) - u_k(x))$$

for all $y \in K$, whenever $m > k$. This implies (u_m) is uniformly Cauchy on K, and thus $u_m \to u$ uniformly on K, as desired. Theorem 1.23 shows that the limit function u is harmonic on Ω.

Now suppose $u(x) = \infty$ for some $x \in \Omega$. Let $y \in \Omega$. Then Harnack's Inequality (3.6), applied to the compact set $K = \{x, y\}$, shows that there is a constant $C \in (1, \infty)$ such that $u_m(x) \leq C u_m(y)$ for every m. Because $u_m(x) \to \infty$, we also have $u_m(y) \to \infty$, and so $u(y) = \infty$. This implies that u is identically ∞ on Ω. ∎

Isolated Singularities

In this section we prove Bôcher's Theorem, which characterizes the behavior of positive harmonic functions in the neighborhood of an isolated singularity. Recall that when $n = 2$, the function $\log(1/|x|)$ is positive and harmonic on $B \setminus \{0\}$, while when $n > 2$, the function $|x|^{2-n}$ is positive and harmonic on $B \setminus \{0\}$. Roughly speaking, Bôcher's Theorem says that near an isolated singularity, a positive harmonic function must behave like one of these functions.

3.9 Bôcher's Theorem: *If u is positive and harmonic on $B \setminus \{0\}$, then there is a function v harmonic on B and a constant $b \geq 0$ such that*

$$u(x) = \begin{cases} v(x) + b\log(1/|x|) & \text{if } n = 2 \\ v(x) + b|x|^{2-n} & \text{if } n > 2 \end{cases}$$

for all $x \in B \setminus \{0\}$.

The next three lemmas will be used in the proof of Bôcher's Theorem (our proof of Bôcher's Theorem is taken from [3], which also contains references to several other proofs of this result). The first lemma describes the spherical averages of a function harmonic on a

punctured ball. Given a continuous function u defined on $B \setminus \{0\}$, we define $A[u](x)$ to be the average of u over the sphere of radius $|x|$:

$$A[u](x) = \int_S u(|x|\zeta)\, d\sigma(\zeta)$$

for $x \in B \setminus \{0\}$.

3.10 Lemma: *Suppose u is harmonic on $B \setminus \{0\}$. Then there exist constants $a, b \in \mathbf{C}$ such that*

$$A[u](x) = \begin{cases} a + b\log(1/|x|) & \text{if } n = 2 \\ a + b|x|^{2-n} & \text{if } n > 2 \end{cases}$$

for all $x \in B \setminus \{0\}$. In particular, $A[u]$ is harmonic on $B \setminus \{0\}$.

PROOF: Let ds denote surface-area measure (unnormalized). Define f on $(0, 1)$ by

$$f(r) = \int_S u(r\zeta)\, ds(\zeta);$$

so $A[u](x)$ is a constant multiple of $f(|x|)$. Because u is continuously differentiable on $B \setminus \{0\}$, we can compute f' by differentiating under the integral sign, obtaining

$$f'(r) = \int_S \zeta \cdot (\nabla u)(r\zeta)\, ds(\zeta) = r^{-n} \int_{rS} \tau \cdot (\nabla u)(\tau)\, ds(\tau).$$

Suppose $0 < r_0 < r_1 < 1$ and $\Omega = \{x \in \mathbf{R}^n : r_0 < |x| < r_1\}$. The divergence theorem, applied to ∇u, shows that

$$\int_{\partial\Omega} \mathbf{n} \cdot \nabla u\, ds = \int_\Omega \Delta u\, dV;$$

here $\mathbf{n}$ denotes the outward unit normal on Ω. Because u is harmonic on Ω, the right side of this equation is 0. Note also that $\partial\Omega = r_0S \cup r_1S$ and that $\mathbf{n} = -\tau/r_0$ on r_0S and $\mathbf{n} = \tau/r_1$ on r_1S. Thus the last equation above implies that

$$\frac{1}{r_0}\int_{r_0S} \tau \cdot (\nabla u)(\tau)\, ds(\tau) = \frac{1}{r_1}\int_{r_1S} \tau \cdot (\nabla u)(\tau)\, ds(\tau),$$

which means $f'(r)$ is a constant multiple of r^{1-n} (for $0 < r < 1$). This proves $f(r)$ is of the desired form. ∎

An immediate consequence of the lemma above is that every radial harmonic function on $B \setminus \{0\}$ is of the form given by the conclusion of 3.10 (a function is *radial* if its value at x depends only on $|x|$). Proof: if u is radial, then $u = A[u]$. (For another proof, see Exercise 13 of this chapter.)

The next lemma is a version of Harnack's Inequality that allows x and y to range over a noncompact set provided $|x| = |y|$.

3.11 Lemma: *There exists a constant $c > 0$ such that for every positive harmonic function u on $B \setminus \{0\}$,*

$$cu(y) < u(x)$$

whenever $0 < |x| = |y| \le 1/2$.

PROOF: Harnack's Inequality (3.6), with $\Omega = B \setminus \{0\}$ and $K = (1/2)S$, shows there is a constant $c > 0$ such that for all positive harmonic u on $B \setminus \{0\}$, we have $cu(y) < u(x)$ whenever $|x| = |y| = 1/2$. Applying this result to the dilates $u_r, 0 < r < 1$, gives the desired conclusion. ∎

The following result characterizes the positive harmonic functions on $B \setminus \{0\}$ that are identically zero on S. This is really the heart of our proof of Bôcher's Theorem.

3.12 Lemma: *Suppose u is positive and harmonic on $B \setminus \{0\}$ and $u(x) \to 0$ as $|x| \to 1$. Then there exists a constant $b > 0$ such that*

$$u(x) = \begin{cases} b\log(1/|x|) & \text{if } n = 2 \\ b|x|^{2-n} & \text{if } n > 2 \end{cases}$$

for all $x \in B \setminus \{0\}$.

PROOF: By Lemma 3.10, we need only show that $u = A[u]$ on $B \setminus \{0\}$. Suppose we could show that $u \ge A[u]$ on $B \setminus \{0\}$. Then if there were a point $x \in B \setminus \{0\}$ such that $u(x) > A[u](x)$, we would have

$$A[u](x) > A[A[u]](x) = A[u](x),$$

a contradiction. Thus we need only prove that $u \ge A[u]$ on $B \setminus \{0\}$, which we now do.

Let c be the constant of Lemma 3.11. By Lemma 3.10, $u - cA[u]$ is harmonic on $B \setminus \{0\}$. By Lemma 3.11, $u(x) - cA[u](x) > 0$ whenever $0 < |x| \le 1/2$, and clearly $u(x) - cA[u](x) \to 0$ as $|x| \to 1$ by our hypothesis on u. The minimum principle for harmonic functions (1.10) thus shows that $u - cA[u] > 0$ on $B \setminus \{0\}$.

We wish to iterate this result. For this purpose, define

$$g(t) = c + t(1 - c)$$

for $t \in [0, 1]$. Suppose we know that

3.13 $$w = u - tA[u] > 0$$

on $B \setminus \{0\}$ for some $t \in [0, 1]$. Since $w(x) \to 0$ as $|x| \to 1$, the preceding argument may be applied to w, yielding

$$w - cA[w] = u - g(t)A[u] > 0$$

on $B \setminus \{0\}$. This process may be continued. Letting $g^{(m)}$ denote the mth iterate of g, we see that 3.13 implies

$$u - g^{(m)}(t)A[u] > 0$$

on $B \setminus \{0\}$ for $m = 1, 2, \ldots$. But $g^{(m)}(t) \to 1$ as $m \to \infty$ for every $t \in [0, 1]$, so that 3.13 holding for some $t \in [0, 1]$ implies $u - A[u] \ge 0$ on $B \setminus \{0\}$. Since 3.13 obviously holds when $t = 0$, we have $u - A[u] \ge 0$ on $B \setminus \{0\}$; as desired. ■

Now we are ready to prove Bôcher's Theorem (3.9).

PROOF OF BÔCHER'S THEOREM: We first assume that $n > 2$ and that u is positive and harmonic on $\overline{B} \setminus \{0\}$. Define a harmonic function w on $B \setminus \{0\}$ by

$$w(x) = u(x) - P[u|_S](x) + |x|^{2-n} - 1.$$

As $|x| \to 1$, we have $w(x) \to 0$ (by 1.17), and as $|x| \to 0$, we have $w(x) \to \infty$ (because u is positive and $P[u|_S]$ is bounded on $B \setminus \{0\}$). By the minimum principle (1.10), we conclude that w is positive on $B \setminus \{0\}$.

Lemma 3.12, applied to w, shows that $u(x) = v(x) + b|x|^{2-n}$ on $B \setminus \{0\}$ for some v harmonic on B and some constant b. Now letting $x \to 0$, we see that the positivity of u implies that $b \geq 0$; we have thus proved Bôcher's Theorem in the case where u is positive and harmonic on $\overline{B} \setminus \{0\}$.

For the general positive harmonic u on $B \setminus \{0\}$, we may apply the result above to the dilate $u(x/2)$, so that

$$u(x/2) = v(x) + b|x|^{2-n}$$

on $B \setminus \{0\}$ for some v harmonic on B and some constant $b \geq 0$. This implies that

$$u(x) = v(2x) + b2^{2-n}|x|^{2-n}$$

on $(1/2)B \setminus \{0\}$, which shows that $u(x) - b2^{2-n}|x|^{2-n}$ extends harmonically to $(1/2)B$, and hence to B. Thus the proof of Bôcher's Theorem is complete in the case where $n > 2$. The proof of the $n = 2$ case is the same, except that $\log(1/|x|)$ should be replaced by $|x|^{2-n}$. ∎

In Chapter 9, in the section *Bôcher's Theorem Revisted*, we will see another approach to this result.

We conclude this section by characterizing the positive harmonic functions on $\mathbf{R}^n \setminus \{0\}$ for $n > 2$. (Recall that by 3.3, a positive harmonic function on $\mathbf{R}^2 \setminus \{0\}$ is constant.)

3.14 Corollary: *Suppose $n > 2$. If u is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$, then there exist constants $a, b \geq 0$ such that*

$$u(x) = a + b|x|^{2-n}$$

for all $x \in \mathbf{R}^n \setminus \{0\}$.

PROOF: Suppose u is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$. Then on $B \setminus \{0\}$ we may write

$$u(x) = v(x) + b|x|^{2-n}$$

as in Bôcher's Theorem (3.9). The function v extends harmonically to all of $\mathbf{R}^n$ by setting $v(x) = u(x) - b|x|^{2-n}$ for $x \in \mathbf{R}^n \setminus B$. Because $\liminf_{x \to \infty} v(x) \geq 0$, the minimum principle (1.10) implies that v is nonnegative on $\mathbf{R}^n$. By 3.1, v is constant, completing the proof. ∎

Positive Harmonic Functions on the Ball

At the end of Chapter 2 we briefly discussed how it is possible to define $P[f]$ when f is not continuous, and indicated that it was necessary to do so in order to characterize the bounded harmonic functions on B. A similar idea works for positive harmonic functions on B—given a positive finite Borel measure μ on S, we can define

$$P[\mu](x) = \int_S P(x, \zeta)\, d\mu(\zeta)$$

for $x \in B$. The function so defined is positive and harmonic on B, as the reader can check by differentiating under the integral sign or by using the converse to the mean-value property. In Chapter 6 we will show (see 6.15) that every positive harmonic function on B is the Poisson integral of a measure as above. Many important consequences follow from this characterization, among them the result (see 6.44) that every positive harmonic function on B has boundary values almost everywhere on S, in a sense to be made precise in Chapter 6.

Exercises

1. Use 3.5 to give another proof of Liouville's Theorem for positive harmonic functions (3.1).

2. Can equality hold in either of the inequalities in Harnack's Inequality for the ball (3.4)?

3. Show that for every multi-index α there exists a constant C_α such that
$$|D^\alpha u(x)| \le \frac{C_\alpha u(0)}{(1-|x|)^{|\alpha|+n-1}}$$
for every $x \in B$ and every positive harmonic u on B. Use this to give another proof of Liouville's Theorem for positive harmonic functions.

4. Let Ω be an open square in $\mathbf{R}^2$. Prove that there exists a positive harmonic function u on Ω such that $u(z)d(z, \partial\Omega)$ is unbounded on Ω.

5. Define s on $\Omega \times \Omega$ by
$$s(x, y) = \sup\{u(y)/u(x) : u \text{ is positive and harmonic on } \Omega\}.$$
Prove that s is continuous on $\Omega \times \Omega$.

6. Suppose u is positive and harmonic on the upper half-space H. Prove that if $z \in H$ and u is bounded on the ray $\{rz : r > 0\}$, then u is bounded in the cone $\Gamma_\alpha(0)$ for every $\alpha > 0$.

7. Suppose u is positive and harmonic on the upper half-space H, $z \in H$, and $u(rz) \to L$ as $r \to 0$, where $L \in [0, \infty]$. Show that if $L = \infty$, then u has nontangential limit ∞ at 0. Prove a similar result for the case $L = 0$. Show that u need not have a nontangential limit at 0 if $L \in (0, \infty)$.

8. Prove the analogue of Theorem 2.10 for positive harmonic functions on H_2 with a common limit along two distinct rays.

9. Suppose u is positive and harmonic on H. Show that u has nontangential limit L at 0 if and only if $\lim_{r\to 0} u(rz) = L$ for every $z \in H$.

10. Show that if pointwise divergence to ∞ occurs in Harnack's Principle (3.8), then the divergence is "uniform" on compact subsets of Ω.

11. Prove that a pointwise convergent sequence of positive harmonic functions on Ω converges uniformly on compact subsets of Ω.

12. Suppose Ω is connected and (u_m) is a sequence of positive harmonic functions on Ω. Show that at least one of the following statements is true:

    (a) (u_m) contains a subsequence diverging to ∞ pointwise on Ω;

    (b) (u_m) contains a subsequence converging uniformly on compact subsets of Ω.

13. Suppose that u is a radial function in $C^2(B \setminus \{0\})$. Let g be the function on $(0, 1)$ defined by $g(|x|) = u(x)$. Compute Δu in terms of g and its derivatives. Use this to prove that a radial harmonic function on $B \setminus \{0\}$ must be of the form given by the conclusion of 3.10.

14. Prove that the constant b and the function v in the conclusion of Bôcher's Theorem (3.9) are unique.

15. Suppose $n > 2$. Assume $a \in \Omega$ and u is harmonic on $\Omega \setminus \{a\}$. Show that if u is positive on some punctured ball centered at a, then there exists a nonnegative constant b and a harmonic function v on Ω such that $u(x) = b|x - a|^{2-n} + v(x)$ on $\Omega \setminus \{a\}$.

16. (a) Suppose $n > 2$. Let u be harmonic on $B \setminus \{0\}$. Show that if

    $$\liminf_{x \to 0} u(x)|x|^{n-2} > -\infty,$$

    then there exists a function v harmonic on B and a constant b such that $u(x) = b|x|^{2-n} + v(x)$ on B.

    (b) Formulate and prove a similar result for $n = 2$.

17. Let $A = \{a_1, a_2, \dots\}$ denote a discrete subset of $\mathbf{R}^n$. Characterize the positive harmonic functions on $\mathbf{R}^n \setminus A$.

CHAPTER 4

The Kelvin Transform

The Kelvin transform performs a role in harmonic function theory analogous to that played by the transformation $f(z) \mapsto f(1/z)$ in holomorphic function theory. For example, it transforms a function harmonic inside the unit sphere into a function harmonic outside the sphere. In this chapter, we introduce the Kelvin transform and use it to solve the Dirichlet problem for the exterior of the unit sphere and to obtain a reflection principle for harmonic functions. Later, we will use the Kelvin transform in the study of isolated singularities of harmonic functions.

Inversion in the Unit Sphere

When studying harmonic functions on unbounded open sets, we will often find it convenient to append the point ∞ to $\mathbf{R}^n$. We topologize $\mathbf{R}^n \cup \{\infty\}$ in the natural way: a set $\omega \subset \mathbf{R}^n \cup \{\infty\}$ is open if it is an open subset of $\mathbf{R}^n$ in the ordinary sense or if $\omega = \{\infty\} \cup (\mathbf{R}^n \setminus E)$, where E is a compact subset of $\mathbf{R}^n$. The resulting topological space is compact and is called the *one-point compactification* of $\mathbf{R}^n$. Via the usual stereographic projection, $\mathbf{R}^n \cup \{\infty\}$ is homeomorphic to the unit sphere in $\mathbf{R}^{n+1}$.

The map $x \mapsto x^*$, where

$$x^* = \begin{cases} x/|x|^2 & \text{if } x \neq 0, \infty \\ 0 & \text{if } x = \infty \\ \infty & \text{if } x = 0 \end{cases}$$

is called the *inversion* of $\mathbf{R}^n \cup \{\infty\}$ relative to the unit sphere. Note that if $x \notin \{0, \infty\}$, then x^* lies on the ray from the origin determined by x, with $|x^*| = 1/|x|$. The reader should verify that the inversion map is continuous, is its own inverse, is the identity on S, and takes a neighborhood of ∞ onto a neighborhood of 0.

For any set $E \subset \mathbf{R}^n \cup \{\infty\}$, we define $E^* = \{x^* : x \in E\}$.

The inversion map preserves the family of spheres and hyperplanes in $\mathbf{R}^n$ (if we adopt the convention that the point ∞ belongs to every hyperplane). To see this, observe that a set $E \subset \mathbf{R}^n$ is a nondegenerate sphere or hyperplane if and only if

4.1 $$E = \{x \in \mathbf{R}^n : a|x|^2 + b \cdot x + c = 0\},$$

where $b \in \mathbf{R}^n$ and a, c are real numbers satisfying $|b|^2 - 4ac > 0$. We easily see that if E has the form 4.1, then E^* has the same form (with the roles of a and c reversed); inversion therefore preserves the family of spheres and hyperplanes, as claimed.

Recall that a C^1-map $\Psi \colon \Omega \to \mathbf{R}^n$ is said to be *conformal* if it preserves angles between intersecting curves; this happens if and only if the Jacobian $\Psi'(x)$ is a scalar multiple of an orthogonal transformation for each $x \in \Omega$.

4.2 Proposition: *The inversion $x \mapsto x^*$ is conformal on $\mathbf{R}^n \setminus \{0\}$.*

PROOF: Set $\Psi(x) = x^* = x/|x|^2$. Fix $y \in \mathbf{R}^n \setminus \{0\}$. Choose an orthogonal transformation T of $\mathbf{R}^n$ such that $Ty = (|y|, 0, \dots, 0)$. Clearly

$$\Psi = T^{-1} \circ \Psi \circ T,$$

so that $\Psi'(y) = T^{-1} \circ \Psi'(T(y)) \circ T$.

Thus to complete the proof we need only show that $\Psi'(T(y))$, which equals $\Psi'(|y|, 0, \dots, 0)$, is a scalar multiple of an orthogonal transformation. However, a simple calculation, which we leave to the reader, shows that the matrix of $\Psi'(|y|, 0, \dots, 0)$ is diagonal, with $-1/|y|^2$ in the first position and $1/|y|^2$ in the other diagonal positions. Hence the proof of the proposition is complete. ∎

Motivation and Definition

Suppose E is a compact subset of $\mathbf{R}^n$. If u is harmonic on $\mathbf{R}^n \setminus E$, we naturally regard ∞ as an isolated singularity of u. When should we say that u has a removable singularity at ∞? There is an obvious answer when $n = 2$, because here the inversion $x \mapsto x^*$ preserves harmonic functions: if $\Omega \subset \mathbf{R}^2 \setminus \{0\}$ and u is harmonic on Ω, then the function $x \mapsto u(x^*)$ is harmonic on Ω^*. (Note that on $\mathbf{R}^2 = \mathbf{C}$, inversion is the map $z \mapsto 1/\bar{z}$.) When $n = 2$, then, we say that u is harmonic at ∞ provided the function $x \mapsto u(x^*)$ has a removable singularity at 0.

Unfortunately, the inversion map does not preserve harmonic functions when $n > 2$ (consider, for example, $u(x) = |x|^{2-n}$). Nevertheless, there is a transformation involving the inversion that does preserve harmonic functions for all $n \geq 2$; it is called the Kelvin transformation in honor of Lord Kelvin who discovered it in the 1840s [17].

We can guess what this transformation is by applying the symmetry lemma to the Poisson kernel. Fixing $\zeta \in S$, recall that $P(\cdot, \zeta)$ is harmonic on $\mathbf{R}^n \setminus \{\zeta\}$ (1.18). By the symmetry lemma (1.13), we have

$$|x - \zeta| = \big|\, |x|^{-1}x - |x|\zeta \,\big|$$

for all $x \in \mathbf{R}^n \setminus \{0\}$. Applying this to $P(x, \zeta) = (1 - |x|^2)/|x - \zeta|^n$, we easily compute that

4.3 $$P(x, \zeta) = -|x|^{2-n} P(x^*, \zeta)$$

for all $x \in \mathbf{R}^n \setminus \{0, \zeta\}$. The significant fact here is that the right side of 4.3 is a harmonic function of x on $\mathbf{R}^n \setminus \{0, \zeta\}$. Except for the minus sign, the definition of the Kelvin transformation is staring us in the face.

Thus, given a function u defined on a set $E \subset \mathbf{R}^n \setminus \{0\}$, we define the function $K[u]$ on E^* by

$$K[u](x) = |x|^{2-n} u(x^*);$$

the function $K[u]$ is called the *Kelvin transform* of u. Note that when $n = 2$, $K[u](x) = u(x^*)$.

We easily see that $K[K[u]] = u$ for all functions u as above; in other words, the Kelvin transform is its own inverse.

The transform K is also linear—if u, v are functions on E and b, c are constants, then $K[bu + cv] = bK[u] + cK[v]$ on E^*.

The Kelvin transform preserves uniform convergence on compact sets. Specifically, suppose E is a compact subset of $\mathbf{R}^n \setminus \{0\}$ and (u_m) is a sequence of functions on E. Then (u_m) converges uniformly on E if and only if $(K[u_m])$ converges uniformly on E^*.

The Kelvin Transform Preserves Harmonic Functions

In this section we will see that the Kelvin transform of every harmonic function is harmonic. We begin with a simple computation.

4.4 Lemma: *If p is a polynomial on $\mathbf{R}^n$ homogeneous of degree m, then*

$$\Delta(|x|^{2-n-2m}p) = |x|^{2-n-2m}\Delta p.$$

PROOF: Let $t \in \mathbf{R}$. Use the product rule for Laplacians (1.19) along with Exercise 3 in Chapter 1 to get

$$\Delta(|x|^t p) = |x|^t \Delta p + 2t|x|^{t-2} x \cdot \nabla p + t(t+n-2)|x|^{t-2}p.$$

If p is homogeneous of degree m, then $x \cdot \nabla p = mp$ (see Exercise 29 in Chapter 1), so the equation above reduces to

4.5 $$\Delta(|x|^t p) = |x|^t \Delta p + t(2m+t+n-2)|x|^{t-2}p.$$

Taking $t = 2 - n - 2m$ now gives the conclusion of the lemma. ∎

If p is homogeneous of degree m, then clearly $K[p] = |x|^{2-n-2m}p$. This observation is used twice in the proof of the next proposition, which shows that the Kelvin transform comes close to commuting with the Laplacian.

4.6 Proposition: *If u is a C^2 function on an open subset of $\mathbf{R}^n \setminus \{0\}$, then*

$$\Delta(K[u]) = K[|x|^4 \Delta u].$$

PROOF: First suppose that p is a polynomial on $\mathbf{R}^n$ homogeneous of degree m. Then

$$\begin{aligned}\Delta(K[p]) &= \Delta(|x|^{2-n-2m}p)\\ &= |x|^{2-n-2m}\Delta p\\ &= K[|x|^4\Delta p],\end{aligned}$$

where the second equality follows from Lemma 4.4 and the third equality holds because $|x|^4\Delta p$ is homogeneous of degree $m+2$.

The paragraph above shows that the proposition holds for polynomials (by linearity). Because polynomials are locally dense in the C^2-norm, the result holds for arbitrary C^2 functions u, as desired. ∎

We come now to the the crucial property of the Kelvin transform.

4.7 Theorem: *If $\Omega \subset \mathbf{R}^n \setminus \{0\}$, then u is harmonic on Ω if and only if $K[u]$ is harmonic on Ω^*.*

PROOF: From the previous proposition, we see that $\Delta(K[u]) \equiv 0$ if and only if $\Delta u \equiv 0$. ∎

Harmonicity at Infinity

Because the Kelvin transform preserves harmonicity, we make the following definition: if $E \subset \mathbf{R}^n$ is compact and u is harmonic on $\mathbf{R}^n \setminus E$, then u is *harmonic* at ∞ provided $K[u]$ has a removable singularity at the origin. Notice that in the $n = 2$ case this definition is consistent with our previous discussion.

If u is harmonic at ∞, then $K[u]$ has a finite limit L at 0; in other words

$$\lim_{x\to 0} |x|^{2-n}u(x/|x|^2) = L.$$

From this we see that if u is harmonic at ∞, then $\lim_{x\to\infty} u(x) = 0$ when $n > 2$, while $\lim_{x\to\infty} u(x) = L$ when $n = 2$. This observation leads to characterizations of harmonicity at ∞. We begin with the $n > 2$ case.

4.8 Theorem: *Assume $n > 2$. Suppose u is harmonic on $\mathbf{R}^n \setminus E$, where $E \subset \mathbf{R}^n$ is compact. Then u is harmonic at ∞ if and only if* $\lim_{x \to \infty} u(x) = 0$.

PROOF: We have just noted above that if u is harmonic at ∞, then $\lim_{x \to \infty} u(x) = 0$.

To prove the other direction, suppose that $\lim_{x \to \infty} u(x) = 0$. Then $|x|^{n-2} K[u](x) \to 0$ as $x \to 0$. By Exercise 2(a) of Chapter 2, $K[u]$ has a removable singularity at 0, which means u is harmonic at ∞. ■

Now we turn to the characterization of harmonicity at ∞ in the $n = 2$ case.

4.9 Theorem: *Suppose u is harmonic on $\mathbf{R}^2 \setminus E$, where $E \subset \mathbf{R}^2$ is compact. Then the following are equivalent:*

(a) *u is harmonic at ∞;*

(b) $\lim_{x \to \infty} u(x) = L$ *for some complex number L;*

(c) $u(x)/\log|x| \to 0$ *as $x \to \infty$;*

(d) *u is bounded on a deleted neighborhood of ∞.*

PROOF: We have already seen that (a) implies (b).

That (b) implies (c) is trivial.

Suppose now that (c) holds. Then $K[u](x)/\log|x| \to 0$ as $x \to 0$. By Exercise 2(b) of Chapter 2, $K[u]$ has a removable singularity at 0. Thus u is harmonic at ∞, which implies (d).

Finally, suppose that (d) holds, so that u is bounded on a deleted neighborhood of ∞. Then $K[u](x) = u(x^*)$ is bounded on a deleted neighborhood of 0. Thus by Theorem 2.3, (a) holds, completing the proof. ■

Boundedness near ∞ is thus equivalent to harmonicity at ∞ when $n = 2$, but not when $n > 2$. We now take up the question of boundedness near ∞ in higher dimensions.

4.10 Theorem: *Suppose $n > 2$ and u is harmonic and real valued on $\mathbf{R}^n \setminus E$, where E is compact. Then the following are equivalent:*

(a) *u is bounded in a deleted neighborhood of ∞;*

(b) *u is bounded above or below in a deleted neighborhood of ∞;*

(c) *$u - c$ is harmonic at ∞ for some constant c;*

(d) *u has a finite limit at ∞.*

PROOF: The implications (a) $\Rightarrow$ (b) and (d) $\Rightarrow$ (a) are trivial. If (c) holds then u has limit c at ∞ by Theorem 4.8; hence (c) $\Rightarrow$ (d). We complete the proof by showing (b) $\Rightarrow$ (c).

Without loss of generality, we assume that u is positive in a deleted neighborhood of ∞. Thus the Kelvin transform $K[u]$ is positive in a deleted neighborhood of 0. By Bôcher's Theorem (3.9) there is a constant c such that $K[u](x) - c|x|^{2-n}$ extends harmonically across 0. Applying the Kelvin transform shows that $u - c$ is harmonic at ∞. ■

Conditions (a), (c), and (d) of Theorem 4.10 are equivalent without the hypothesis that u is real valued.

Note that Theorem 4.10 provides a new proof of Liouville's Theorem for positive harmonic functions (3.1). Specifically, if $n > 2$ and u is positive and harmonic on $\mathbf{R}^n$, then by Theorem 4.10 u must have finite limit c at ∞. By the maximum/minimum principle, $u \equiv c$. This new proof of Liouville's Theorem amounts to the observation that—via the Kelvin transform—Bôcher's Theorem implies Liouville's Theorem, at least for $n >.2$.

The implication also holds when $n = 2$. If u is positive and harmonic on $\mathbf{R}^2$, then by Bôcher's Theorem (3.9) there is a constant $b \geq 0$ such that $v(x) = K[u](x) - b \log |1/x|$ has harmonic extension across 0. Thus v is an entire harmonic function. If $b > 0$, then we would have $\lim_{x\to\infty} v(x) = \infty$, which contradicts the minimum principle. Thus $b = 0$ and $\lim_{x\to\infty} v(x) = u(0)$, from which it follows that $v \equiv u(0)$. Hence $u \equiv u(0)$, as desired.

In Chapter 9, we will see that Liouville's Theorem implies Bôcher's Theorem when $n > 2$, and we will present generalized versions of these theorems.

The Exterior Dirichlet Problem

In Chapter 1, we solved the Dirichlet problem for the interior of the unit sphere S—given any $f \in C(S)$, there is a unique function u harmonic on B and continuous on $\overline{B}$ such that $u|_S = f$. To solve the corresponding problem for the exterior of the unit sphere, we define the *exterior Poisson kernel*, denoted P_e, by setting

$$P_e(x, \zeta) = \frac{|x|^2 - 1}{|x - \zeta|^n}$$

for $|x| > 1$ and $\zeta \in S$. Given $f \in C(S)$, we define the exterior Poisson integral $P_e[f]$ by

$$P_e[f](x) = \int_S f(\zeta) P_e(x, \zeta)\, d\sigma(\zeta)$$

for $|x| > 1$.

4.11 Theorem: *Suppose $f \in C(S)$. Then there is a unique function u harmonic on B^* and continuous on $\overline{B}^*$ such that $u|_S = f$. Moreover, $u = P_e[f]$ on $B^* \setminus \{\infty\}$.*

REMARK: For $n > 2$, we are not asserting that there exists a unique continuous u on B^*, with u harmonic on $\{x \in \mathbf{R}^n : |x| > 1\}$, such that $u|_S = f$. For example, the functions $1 - |x|^{2-n}$ and 0, which agree on S, are both harmonic on $\mathbf{R}^n \setminus \{0\}$. The uniqueness assertion in the theorem above comes from the requirement that u be harmonic at ∞ (recall that $\infty \in B^*$).

PROOF OF THEOREM 4.11: Let $v \in C(\overline{B})$ denote the solution of the Dirichlet problem for B with boundary data f on S, so that $v|_S = f$ and

$$v(x) = \int_S f(\zeta) P(x, \zeta)\, d\sigma(\zeta)$$

for $x \in B$. The function $u = K[v]$ is then harmonic on B^* (if we set $u(\infty) = \lim_{x \to \infty} K[v](x)$), u is continuous on $\overline{B}^*$, and $u|_S = f$.

We have

$$u(x) = \int_S f(\zeta) |x|^{2-n} P(x^*, \zeta)\, d\sigma(\zeta)$$

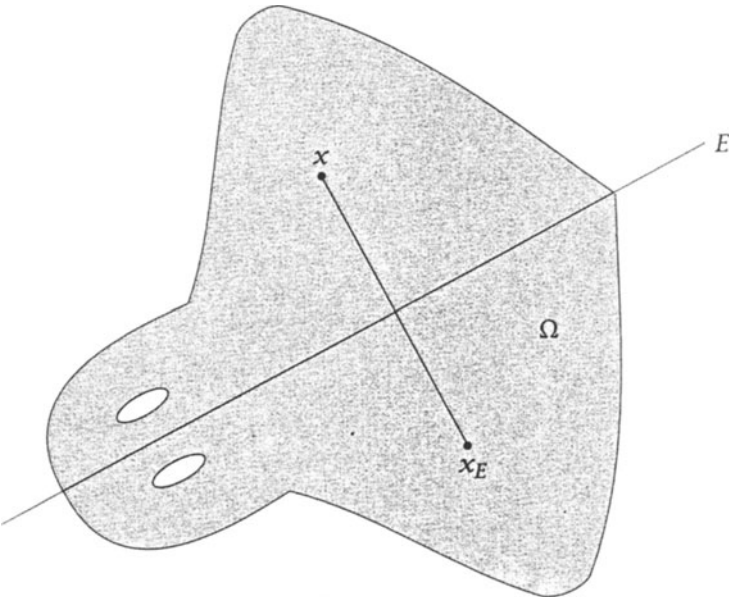
for $|x| > 1$. By 4.3, this gives $u = P_e[f]$ on $B^* \setminus \{\infty\}$, as desired.

The uniqueness of u follows from the maximum principle. ■

Symmetry and the Schwarz Reflection Principle

Given a hyperplane E, we say that a pair of points are *symmetric* about E if E is the perpendicular bisector of the line segment joining these points. For each $x \in \mathbf{R}^n$, there exists a unique $x_E \in \mathbf{R}^n$ such that x and x_E are symmetric about E; we call x_E the *reflection* of x in E. Clearly $(x_E)_E = x$ for every $x \in \mathbf{R}^n$.

We say Ω is *symmetric* about the hyperplane E if $\Omega_E = \Omega$, where $\Omega_E = \{x_E : x \in \Omega\}$.



Ω is symmetric about E.

If T is a translation, dilation, or rotation, then T preserves symmetry about hyperplanes; in other words, if T is any of these maps and E is a hyperplane, then $T(x)$ and $T(x_E)$ are symmetric about $T(E)$ for all $x \in \mathbf{R}^n$.

Given a hyperplane $E = \{x \in \mathbf{R}^n : b \cdot x = c\}$, where b is a nonzero vector in $\mathbf{R}^n$ and c is a real number, we set $E^+ = \{x \in \mathbf{R}^n : b \cdot x > c\}$; geometrically, E^+ is an open half-space with $\partial E^+ = E$.

We now come to the Schwarz reflection principle for hyperplanes; the reader who has done Exercise 3 in Chapter 2 can probably guess the proof.

4.12 Theorem: *Suppose Ω is symmetric about a hyperplane E. If u is continuous on $\Omega \cap \overline{E^+}$, u is harmonic on $\Omega \cap E^+$, and $u = 0$ on $\Omega \cap E$, then u extends harmonically to Ω.*

PROOF: We may assume that $E = \{(x, y) \in \mathbf{R}^n : y = 0\}$ and that $E^+ = \{(x, y) \in \mathbf{R}^n : y > 0\}$. The function

$$v(x,y) = \begin{cases} u(x,y) & \text{if } (x,y) \in \Omega \text{ and } y \geq 0 \\ -u(x,-y) & \text{if } (x,y) \in \Omega \text{ and } y < 0 \end{cases}$$

is continuous on Ω and satisfies the mean-value property. Hence, by Theorem 1.24, v is a harmonic extension of u to all of Ω. ■

We now extend the notions of symmetry and reflection to spheres. If $E = S$, the unit sphere, then inversion is the natural choice for the reflection map $x \mapsto x_E$. So here we set $x_E = x^*$. More generally, if $E = \partial B(a, r)$, we define

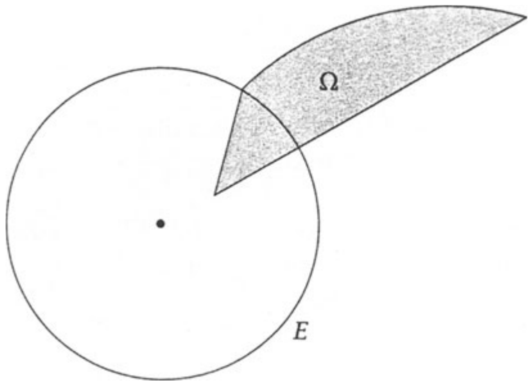
4.13 $$x_E = a + r^2(x - a)^*,$$

and we say that x and x_E are *symmetric* about E. Note that the center of E and the point at infinity are symmetric about E. We say that Ω is *symmetric* about the sphere E if $\Omega_E = \Omega$, where $\Omega_E = \{x_E : x \in \Omega\}$; see 4.14.

We remark in passing that symmetry about a hyperplane can be viewed as a limiting case of symmetry about a sphere; see Exercise 10 of this chapter. (We adopt the convention that $\infty_E = \infty$ when E is a hyperplane.)

Translations, dilations, and rotations obviously preserve symmetry about spheres. The inversion map also preserves symmetry—about spheres as well as hyperplanes—although this is far from obvious. Let us look at a special case we need below. Suppose E is the sphere with center $(0, \ldots, 0, 1)$ and radius 1. Then E contains the origin, so that E^* is a hyperplane; in fact,

$$E^* = \{(x, y) \in \mathbf{R}^n : y = 1/2\} \cup \{\infty\},$$



4.14 *Ω is symmetric about E.*

as the reader can easily check. Assume for the moment that $n = 2$; here we identify $\mathbf{R}^2$ with $\mathbf{C}$, so that inversion is the map $z \mapsto 1/\overline{z}$. Given $z \in \mathbf{C} \setminus \{0\}$, we need to show that z^* and $(z_E)^*$ are symmetric about E^*. A moment's reflection shows that to do this we need only verify that the conjugate of $z^* - i/2$ equals $(z_E)^* - i/2$; this bit of algebra we leave to the reader. To go from $\mathbf{R}^2$ to $\mathbf{R}^n$ with $n > 2$, observe that inversion preserves every linear subspace of $\mathbf{R}^n$. Given $z \in \mathbf{R}^n \setminus \{0\}$, then, we look at the two-dimensional plane determined by 0, z, and z_E. Because the center of E is on the line determined by z and z_E, this plane contains the $(0, y)$-axis. We can thus view this plane as $\mathbf{C}$, with $(0, \dots, 0, 1)$ playing the role of i. The proof for $\mathbf{R}^2$ therefore shows that z^* and $(z_E)^*$ are symmetric about E^* in $\mathbf{R}^n$.

We can now prove the Schwarz reflection principle for regions symmetric about spheres.

4.15 Theorem: *Suppose Ω is a region symmetric about $\partial B(a, r)$. If u is continuous on $\Omega \cap \overline{B}(a, r)$, u is harmonic on $\Omega \cap B(a, r)$, and $u = 0$ on $\Omega \cap \partial B(a, r)$, then u extends harmonically to Ω.*

PROOF: We may assume $a = (0, \dots, 0, 1)$ and $r = 1$; we are then dealing with the sphere E discussed above. Because Ω is symmetric about E, Ω^* is symmetric about the hyperplane E^*, as we just showed. Our hypotheses on u now show that the Schwarz reflection principle for hyperplanes (4.12) can be applied to the Kelvin transform of u. Accordingly, $K[u]$ extends to a function v harmonic on Ω^*. Because

the Kelvin transform is its own inverse, $K[v]$ extends u harmonically to Ω. ■

Let us explicitly identify the harmonic reflection of u across a sphere in the concrete case of S, the unit sphere.

4.16 Theorem: *Suppose Ω is connected and symmetric about S. If u is continuous on $\Omega \cap \overline{B}$, u is harmonic on $\Omega \cap B$, and $u = 0$ on $\Omega \cap S$, then the function v defined on Ω by*

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega \cap \overline{B} \\ -K[u](x) & \text{if } x \in \Omega \cap (\mathbf{R}^n \setminus \overline{B}) \end{cases}$$

is the unique harmonic extension of u to Ω.

PROOF: Set $a = (0, \ldots, 0, 1)$ and define $w(x) = v(x - a)$; the domain of the function w is then $\Omega + a$, which is symmetric about the sphere E of the previous proof. We will be done if we can show that $K[w]$ has the appropriate reflection property about the hyperplane E^*. What we need to show, then, is that

$$K[w]((x + a)^*) = -K[w]((x^* + a)^*)$$

for all $x \in \Omega$. This amounts to showing that

$$|x + a|^{n-2} v(x) = -|x^* + a|^{n-2} v(x^*)$$

for all $x \in \Omega$. By Exercise 1 of this chapter, $|x^* + a|/|x + a| = |x|^{-1}$. We therefore need only show that $v = -K[v]$ on Ω. But this last identity follows easily from the definition of v. ■

Exercises

1. Show that if $\zeta \in S$ and $x \in \mathbf{R}^n \setminus \{0\}$, then
$$|x^* + \zeta| = \frac{|x + \zeta|}{|x|}.$$

2. Show that at $x \in \mathbf{R}^n \setminus \{0\}$, the determinant of the Jacobian of the inversion map equals $-1/|x|^{2n}$.

3. Let f be a function of one complex variable that is holomorphic on the complement of some disk. We say that f is *holomorphic* at ∞ provided $f(1/z)$ has a removable singularity at 0. Show that the following are equivalent:
   (a) f is holomorphic at ∞;
   (b) f is bounded on a deleted neighborhood of ∞;
   (c) $\lim_{z\to\infty} f(z)/z = 0$.

4. Assume $\omega \subset \mathbf{R}^n \cup \{\infty\}$ is open. Show that u is harmonic on ω if and only if $K[u]$ is harmonic on ω^*.

5. (a) Show that if $n > 2$, then the only harmonic function on $\mathbf{R}^n \cup \{\infty\}$ is identically zero.
   (b) Prove that all harmonic functions on $\mathbf{R}^2 \cup \{\infty\}$ are constant.

6. Suppose that u is harmonic and positive on $\mathbf{R}^2 \setminus E$, where E is compact. Characterize the behavior of u near ∞.

7. Prove that the solution to the exterior Dirichlet problem in Theorem 4.11 is unique.

8. Suppose that f is continuous on $\partial B(a,r)$ and that u solves the Dirichlet problem for $B(a,r)$ with boundary data f. What is the solution (expressed in terms of u) of the Dirichlet problem for $(\mathbf{R}^n \cup \{\infty\}) \setminus \overline{B}(a,r)$ with boundary data f?

9. Let E denote the hyperplane $\{x \in \mathbf{R}^n : b \cdot x = c\}$, where b is a nonzero vector in $\mathbf{R}^n$ and c is a real number. For $x \in \mathbf{R}^n$, show that the reflection x_E is given by the formula
$$x_E = x - \frac{2((x \cdot b) - c)b}{|b|^2}.$$

10. Let E denote the hyperplane $\mathbf{R}^{n-1} \times \{0\}$. Fix a point $z = (0, y)$ in the upper half-space. Show that the reflections of z about spheres of radius R centered at $(0, R)$ converge to $z_E = (0, -y)$ as $R \to \infty$.

11. Suppose E is a sphere of radius r centered at a, with $0 \notin E$. Show that the radius of E^* is $r / |r^2 - |a|^2|$ and the center of E^* is $a/(|a|^2 - r^2)$.

12. Show that the inversion map preserves symmetry about spheres and hyperplanes. In other words, if E is a sphere or hyperplane, then x^* and $(x_E)^*$ are symmetric about E^* for all x.

13. Let E be a compact subset of S with nonempty interior relative to S. Prove that there exists a nonconstant bounded harmonic function on $\mathbf{R}^n \setminus E$.

CHAPTER 5

Harmonic Polynomials

Recall the Dirichlet problem for the ball in $\mathbf{R}^n$: given $f \in C(S)$, find $u \in C(\overline{B})$ such that u is harmonic on B and $u|_S = f$. We know from Chapter 1 that

$$u(x) = P[f](x) = \int_S f(\zeta) \frac{1 - |x|^2}{|x - \zeta|^n} \, d\sigma(\zeta)$$

for $x \in B$. To prove that $P[f]$ is harmonic on B, we computed its Laplacian by differentiating under the integral sign in the equation above and noting that for each fixed $\zeta \in S$, the Poisson kernel $(1 - |x|^2)/|x - \zeta|^n$ is harmonic as a function of x.

Suppose now that f is a polynomial on $\mathbf{R}^n$ restricted to S. For fixed $\zeta \in S$, the Poisson kernel $(1 - |x|^2)/|x - \zeta|^n$ is not a polynomial in x, so nothing in the formula above suggests that $P[f]$ should be a polynomial. Thus our first result in this chapter should be somewhat of a surprise: $P[f]$ is indeed a polynomial, and its degree is at most the degree of f.

Further indications of the importance of harmonic polynomials will come when we prove that every polynomial on $\mathbf{R}^n$ can be written as the sum of a harmonic polynomial and a polynomial multiple of $|x|^2$. This result will then be used to decompose the Hilbert space $L^2(S)$ into a direct sum of spaces of harmonic polynomials. As we will see, this decomposition is the higher-dimensional analogue of the Fourier series decomposition of a function on the unit circle in $\mathbf{R}^2$.

Our theory will lead to a fast algorithm for computing the Poisson integral of any polynomial. The algorithm involves differentiation, but not integration!

Next, we will use the Kelvin transform to find an explicit basis for the space of harmonic polynomials that are homogeneous of degree m. The chapter concludes with a study of zonal harmonics, which are used to decompose the Poisson kernel and to show that the homogeneous expansion of a harmonic function has nice convergence properties.

Polynomial Decompositions

We begin with a crucial theorem showing that the Poisson integral of a polynomial is a polynomial of a special form. The proof uses, without comment, the result that the Poisson integral gives the unique solution to the Dirichlet problem.

Note that the theorem below implies that if p is a polynomial, then the degree of $P[p|_S]$ is less than or equal to the degree of p. This inequality can be strict; for example, if $p(x) = |x|^2$, then $P[p|_S] \equiv 1$.

5.1 Theorem: *If p is a polynomial on $\mathbf{R}^n$ of degree m, then*

$$P[p|_S] = (1 - |x|^2)q + p$$

for some polynomial q of degree at most $m - 2$.

PROOF: Let p be a polynomial on $\mathbf{R}^n$ of degree m. If $m = 0$ or $m = 1$, then p is harmonic and hence $P[p|_S] = p$, so the desired result follows by taking $q = 0$. Thus we can assume that $m \geq 2$.

For any choice of q, the function $(1 - |x|^2)q + p$ equals p on S. Thus to solve the Dirichlet problem for B with boundary data $p|_S$, we need only find q such that $(1 - |x|^2)q + p$ is harmonic. In other words, to prove the theorem we need only show that there exists a polynomial q of degree at most $m - 2$ such that

5.2
$$\Delta((1 - |x|^2)q) = -\Delta p.$$

To do this, let W denote the vector space of all polynomials on $\mathbf{R}^n$ of degree at most $m - 2$, and define a linear map $T: W \to W$ by

$$T(q) = \Delta((1 - |x|^2)q).$$

If $T(q) = 0$, then $(1 - |x|^2)q$ is a harmonic function; this harmonic function equals 0 on S, and hence by the maximum principle equals 0 on B; this forces q to be 0. Thus T is injective.

We now use the magic of linear algebra (an injective linear map from a finite-dimensional vector space to itself is also surjective) to conclude that T is surjective. Hence there exists a polynomial q of degree at most $m - 2$ such that 5.2 holds, and we are done. ∎

The following corollary will be a key tool in our proof of the direct-sum decomposition of the polynomials (Proposition 5.5). Here "polynomial" means a polynomial on $\mathbf{R}^n$, and "nonzero" means not identically 0.

5.3 Corollary: *No nonzero polynomial multiple of $|x|^2$ is harmonic.*

PROOF: Suppose p is a nonzero polynomial on $\mathbf{R}^n$ of degree m and $|x|^2 p$ is harmonic. Because $p|_S = (|x|^2 p)|_S$, the Poisson integral $P[p|_S]$ must equal the harmonic polynomial $|x|^2 p$, which has degree $m + 2$. This contradicts the previous theorem, which implies that the degree of $P[p|_S]$ is at most m. ∎

Every polynomial p on $\mathbf{R}^n$ with degree m can be uniquely written in the form $p = \sum_{j=0}^{m} p_j$, where each p_j is a homogeneous polynomial on $\mathbf{R}^n$ of degree j. We call p_j the *homogeneous part* of p of degree j. Note that $\Delta p = \sum_{j=0}^{m} \Delta p_j$, and thus p is harmonic if and only if each p_j is harmonic (because a polynomial is identically 0 if and only if each homogeneous part of it is identically 0).

In the next section we will be working in $L^2(S)$. Two distinct polynomials of the same degree can have equal restrictions to S, but two homogeneous polynomials of the same degree that agree on S must agree everywhere. Thus we will find it convenient to restrict attention to homogeneous polynomials. Let us denote by $\mathcal{P}_m(\mathbf{R}^n)$ the complex vector space of all homogeneous polynomials on $\mathbf{R}^n$ of degree m. Let $\mathcal{H}_m(\mathbf{R}^n)$ denote the subspace of $\mathcal{P}_m(\mathbf{R}^n)$ consisting of all homogeneous harmonic polynomials on $\mathbf{R}^n$ of degree m. For example,

5.4 $$p(x,y,z) = 8x^5 - 40x^3y^2 + 15xy^4 - 40x^3z^2 + 30xy^2z^2 + 15xz^4$$

is an element of $\mathcal{H}_5(\mathbf{R}^3)$, as the reader can verify; we have used (x,y,z) in place of (x_1,x_2,x_3) to denote a typical point in $\mathbf{R}^3$.

In the following proposition, we write $\mathcal{P}_m(\mathbf{R}^n)$ as the algebraic direct sum of the two subspaces $\mathcal{H}_m(\mathbf{R}^n)$ and $|x|^2\mathcal{P}_{m-2}(\mathbf{R}^n)$, meaning that every element of $\mathcal{P}_m(\mathbf{R}^n)$ can be uniquely written as the sum of an element of $\mathcal{H}_m(\mathbf{R}^n)$ and an element of $|x|^2\mathcal{P}_{m-2}(\mathbf{R}^n)$. In the next section we will see that this is an orthogonal decomposition when we restrict all functions to S and use the usual inner product that comes from surface-area measure.

5.5 Proposition: *If $m \geq 2$, then*

$$\mathcal{P}_m(\mathbf{R}^n) = \mathcal{H}_m(\mathbf{R}^n) \bigoplus |x|^2\mathcal{P}_{m-2}(\mathbf{R}^n).$$

PROOF: Let $p \in \mathcal{P}_m(\mathbf{R}^n)$. Then

$$p = P[p|_S] + |x|^2 q - q$$

for some polynomial q of degree at most $m-2$ (by Theorem 5.1). Take the homogeneous part of degree m of both sides of the equation above, getting

5.6 $$p = p_m + |x|^2 q_{m-2},$$

where p_m is the homogeneous part of degree m of the harmonic function $P[p|_S]$ (and hence $p_m \in \mathcal{H}_m(\mathbf{R}^n)$) and q_{m-2} is the homogeneous part of degree $m-2$ of q (and hence $q_{m-2} \in \mathcal{P}_{m-2}(\mathbf{R}^n)$). Thus every element of $\mathcal{P}_m(\mathbf{R}^n)$ can be written as the sum of an element of $\mathcal{H}_m(\mathbf{R}^n)$ and an element of $|x|^2\mathcal{P}_{m-2}(\mathbf{R}^n)$.

To show that this decomposition is unique, suppose that

$$p_m + |x|^2 q_{m-2} = \tilde{p}_m + |x|^2 \tilde{q}_{m-2},$$

where $p_m, \tilde{p}_m \in \mathcal{H}_m(\mathbf{R}^n)$ and $q_{m-2}, \tilde{q}_{m-2} \in \mathcal{P}_{m-2}(\mathbf{R}^n)$. Then

$$p_m - \tilde{p}_m = |x|^2(\tilde{q}_{m-2} - q_{m-2}).$$

The left side of the equation above is harmonic, and the right side is a polynomial multiple of $|x|^2$. Thus Corollary 5.3 implies that $p_m = \tilde{p}_m$ and $q_{m-2} = \tilde{q}_{m-2}$, as desired. ■

The map $p \mapsto p_m$, where $p \in \mathcal{P}_m(\mathbf{R}^n)$ and $p_m \in \mathcal{H}_m(\mathbf{R}^n)$ are as in 5.6, is called the *canonical projection of $\mathcal{P}_m(\mathbf{R}^n)$ onto $\mathcal{H}_m(\mathbf{R}^n)$*. Later we will find a formula for this projection (see Theorem 5.18).

We now come to the main result of this section. As usual, $[t]$ denotes the largest integer less than or equal to t. Thus in the theorem below, the last index $m-2k$ equals 0 or 1, depending upon whether m is even or odd.

5.7 Theorem: *Every $p \in \mathcal{P}_m(\mathbf{R}^n)$ can be uniquely written in the form*

$$p = p_m + |x|^2 p_{m-2} + \cdots + |x|^{2k} p_{m-2k},$$

where $k = [\frac{m}{2}]$ and each $p_j \in \mathcal{H}_j(\mathbf{R}^n)$.

PROOF: The desired result obviously holds when $m = 0$ or $m = 1$, because $\mathcal{P}_m(\mathbf{R}^n) = \mathcal{H}_m(\mathbf{R}^n)$ in those cases. Thus we can assume that $m \geq 2$.

Suppose that $p \in \mathcal{P}_m(\mathbf{R}^n)$. By the previous proposition, p can be uniquely written in the form

$$p = p_m + |x|^2 q,$$

where $p_m \in \mathcal{H}_m(\mathbf{R}^n)$ and $q \in \mathcal{P}_{m-2}(\mathbf{R}^n)$. By induction, we can assume that the theorem holds when m is replaced by $m-2$. Taking the unique decomposition for q given by the theorem and plugging it into the equation above gives the desired decomposition of p. This decomposition is unique because p_m is uniquely determined and the decomposition of q is also uniquely determined. ∎

If $p \in \mathcal{P}_m(\mathbf{R}^n)$ and $p_m, p_{m-2}, \ldots, p_{m-2k}$ are as in the theorem above, then the solution to the Dirichlet problem for B with boundary data $p|_S$ is

$$p_m + p_{m-2} + \cdots + p_{m-2k}.$$

To see this, observe that the function above is harmonic and that it agrees with p on S. Later in this chapter we will develop an algorithm for computing $p_m, p_{m-2}, \ldots, p_{m-2k}$ (and thus $P[p|_S]$) from p.

We finish this section by computing $\dim \mathcal{H}_m(\mathbf{R}^n)$, the dimension (over $\mathbf{C}$) of the vector space $\mathcal{H}_m(\mathbf{R}^n)$. Because $\mathcal{H}_0(\mathbf{R}^n)$ is the space of constant functions, $\dim \mathcal{H}_0(\mathbf{R}^n) = 1$. Because $\mathcal{H}_1(\mathbf{R}^n)$ is the space of

linear functions on $\mathbf{R}^n$, we have $\dim \mathcal{H}_1(\mathbf{R}^n) = n$. The next proposition takes care of higher values of m.

5.8 Proposition: *If $m \geq 2$, then*

$$\dim \mathcal{H}_m(\mathbf{R}^n) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}.$$

PROOF: We begin by finding $\dim \mathcal{P}_m(\mathbf{R}^n)$. Because the monomials $\{x^\alpha : |\alpha| = m\}$ form a basis of $\mathcal{P}_m(\mathbf{R}^n)$, $\dim \mathcal{P}_m(\mathbf{R}^n)$ equals the number of distinct multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = m$. Adding 1 to each α_j, we see that $\dim \mathcal{P}_m(\mathbf{R}^n)$ equals the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, with each $\alpha_j > 0$, such that $|\alpha| = n + m$. Now consider removing $n-1$ integers from the interval $(0, n+m) \subset \mathbf{R}$. This partitions $(0, n+m)$ into n disjoint open intervals. Letting $\alpha_1, \dots, \alpha_n$ denote the lengths of these intervals, taken in order, we have

$$\sum_{j=1}^{n} \alpha_j = n + m.$$

Each choice of $n-1$ integers from $(0, n+m)$ thus generates a multi-index α with $|\alpha| = n+m$, and each multi-index of degree $n+m$ arises from one and only one such choice. The number of such choices is, of course, $\binom{n+m-1}{n-1}$. Thus

$$\dim \mathcal{P}_m(\mathbf{R}^n) = \binom{n+m-1}{n-1}.$$

From Proposition 5.5 we have

$$\dim \mathcal{H}_m(\mathbf{R}^n) = \dim \mathcal{P}_m(\mathbf{R}^n) - \dim \mathcal{P}_{m-2}(\mathbf{R}^n).$$

Combining the last two equations gives the desired result. ■

Spherical Harmonic Decomposition of $L^2(S)$

In Proposition 5.5, we showed that the space of homogeneous polynomials of degree m decomposes as the direct sum of the space of

harmonic homogeneous polynomials of degree m and $|x|^2$ times the homogeneous polynomials of degree $m - 2$. Now we turn to ideas revolving around orthogonal direct sums, which means that we need to introduce an inner product.

Because a homogeneous function on $\mathbf{R}^n$ is determined by its restriction to S, we follow the natural impulse to work in $L^2(S, d\sigma)$, which we denote simply by $L^2(S)$. In other words, $L^2(S)$ denotes the usual Hilbert space of Borel-measurable square-integrable functions on S with inner product defined by

$$\langle f, g \rangle = \int_S f \overline{g} \, d\sigma.$$

Our main result in this section will be a natural orthogonal decomposition of $L^2(S)$.

Homogeneous polynomials on $\mathbf{R}^n$ of different degrees, when restricted to S, are not necessarily orthogonal in $L^2(S)$. For example, ${x_1}^2$ and ${x_1}^4$ are not orthogonal in this space because their product is positive everywhere on S. However, the next proposition shows that if the homogeneous polynomial of higher degree is harmonic, then we indeed have orthogonality (because $\mathcal{H}_m(\mathbf{R}^n)$ is closed under complex conjugation).

5.9 Proposition: *If p, q are polynomials on $\mathbf{R}^n$ and q is harmonic and homogeneous with degree higher than the degree of p, then*

$$\int_S pq \, d\sigma = 0.$$

PROOF: The desired conclusion involves only the values of p and q on S. Hence by linearity and Theorem 5.7, it suffices to prove the proposition when p is replaced by a homogeneous harmonic polynomial. Thus we can assume that $p \in \mathcal{H}_k(\mathbf{R}^n)$ and that $q \in \mathcal{H}_m(\mathbf{R}^n)$, where $k < m$.

Green's identity (1.1) implies that

5.10
$$\int_S (pD_{\mathbf{n}}q - qD_{\mathbf{n}}p) \, d\sigma = 0.$$

But for $\zeta \in S$,

$$(D_{\mathbf{n}}p)(\zeta) = \frac{d}{dr} p(r\zeta)|_{r=1} = \frac{d}{dr}(r^k p(\zeta))|_{r=1} = kp(\zeta).$$

Similarly, $D_{\mathbf{n}}q = mq$ on S. Thus 5.10 implies that

$$(m-k)\int_S pq\,d\sigma = 0.$$

Because $k < m$, the last integral vanishes, as desired. ∎

Obviously $|x|^2\mathcal{P}_{m-2}(\mathbf{R}^n)$ restricted to S is the same as $\mathcal{P}_{m-2}(\mathbf{R}^n)$ restricted to S. Thus the last proposition shows that if we restrict all functions to S, then the decomposition given in Proposition 5.5 is an orthogonal decomposition with respect to the inner product on $L^2(S)$.

The restriction of $\mathcal{H}_m(\mathbf{R}^n)$ to S is sufficiently important to receive its own name and notation. A *spherical harmonic* of *degree m* is the restriction to S of an element of $\mathcal{H}_m(\mathbf{R}^n)$. The collection of all spherical harmonics of degree m will be denoted by $\mathcal{H}_m(S)$; thus

$$\mathcal{H}_m(S) = \{p|_S : p \in \mathcal{H}_m(\mathbf{R}^n)\}.$$

The map $p \mapsto p|_S$ provides an identification of the complex vector space $\mathcal{H}_m(\mathbf{R}^n)$ with the complex vector space $\mathcal{H}_m(S)$. We use the notation $\mathcal{H}_m(S)$ when we want to emphasize that we are considering the functions to be defined only on S.

For example, take $n = 3$ and consider the function

5.11 $$q(x,y,z) = 15x - 70x^3 + 63x^5$$

defined for $(x,y,z) \in S$. Is q an element of $\mathcal{H}_5(S)$? Although q appears to be neither harmonic nor homogeneous of degree 5, note that on S we have

$$q(x,y,z) = 15x(x^2+y^2+z^2)^2 - 70x^3(x^2+y^2+z^2) + 63x^5.$$

The right side of the equation above is a homogeneous polynomial on $\mathbf{R}^3$ of degree 5, and as the reader can check, it is harmonic. Thus q, as defined by 5.11, is indeed an element of $\mathcal{H}_5(S)$. (To save a bit of work, note that the right side of the equation above equals the polynomial $p \in \mathcal{H}_5(\mathbf{R}^3)$ defined by 5.4, so $q = p|_S$. Examples 5.4 and 5.11 were generated using the software described in Appendix B.)

Restating some previous results in terms of spherical harmonics, we see that Proposition 5.9 implies that $\mathcal{H}_k(S)$ is orthogonal to $\mathcal{H}_m(S)$ in $L^2(S)$ whenever $k \neq m$. Theorem 5.7 implies that if p is a polynomial

on $\mathbf{R}^n$ of degree m, then $p|_S$ can be written as a sum of spherical harmonics of degree at most m. In our next theorem, we will use these results to decompose $L^2(S)$ into an infinite direct sum of spaces of spherical harmonics.

We will need a bit of Hilbert space theory. Recall that if H is a complex Hilbert space, then we write $H = \bigoplus_{m=0}^{\infty} H_m$ when the following three conditions are satisfied:

(a) H_m is a closed subspace of H for every m.

(b) H_k is orthogonal to H_m if $k \neq m$.

(c) For every $x \in H$, there exist $x_m \in H_m$ such that

$$x = x_0 + x_1 + \cdots,$$

the sum converging in the norm of H.

When (a), (b), and (c) hold, the Hilbert space H is said to be the *direct sum* of the spaces H_m. If this is the case, then the expansion in (c) is unique. Also, if (a) and (b) hold, then (c) holds if and only if the complex linear span of $\bigcup_{m=0}^{\infty} H_m$ is dense in H.

We can now easily prove the main result of this section.

5.12 Theorem: $L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$.

PROOF: Condition (a) above holds because each $\mathcal{H}_m(S)$ is finite dimensional and hence is closed in $L^2(S)$.

We have already noted that condition (b) above follows from Proposition 5.9.

To verify condition (c), we need only show that the linear span of $\bigcup_{m=0}^{\infty} \mathcal{H}_m(S)$ is dense in $L^2(S)$. As we have already noted, Theorem 5.7 implies that if p is a polynomial on $\mathbf{R}^n$, then $p|_S$ can be written as a finite sum of elements of $\bigcup_{m=0}^{\infty} \mathcal{H}_m(S)$. By the Stone-Weierstrass Theorem (see [14], Theorem 7.33), the set of restrictions $p|_S$, as p ranges over all polynomials on $\mathbf{R}^n$, is dense in $C(S)$ with respect to the supremum norm. Because $C(S)$ is dense in $L^2(S)$ and the L^2-norm is less than or equal to the L^∞-norm on S, this implies that the linear span of $\bigcup_{m=0}^{\infty} \mathcal{H}_m(S)$ is dense in $L^2(S)$, as desired. ∎

The theorem above reduces to a familiar result when $n = 2$. To see this, suppose $p \in \mathcal{H}_m(\mathbf{R}^2)$ is real valued. Then p is the real part

of an entire holomorphic function f whose imaginary part vanishes at the origin. The Cauchy-Riemann equations imply that all (complex) derivatives of f except the mth derivative vanish at the origin. Thus $f = cz^m$ for some complex constant c, and so

$$p = cz^m + \overline{cz^m}.$$

This implies that $\mathcal{H}_m(\mathbf{R}^2)$ is the complex linear span of $\{z^m, \overline{z^m}\}$. Thus $\mathcal{H}_m(S)$, as a space of functions of the variable $e^{i\theta}$, is the complex linear span of $\{e^{im\theta}, e^{-im\theta}\}$ (or of $\{\cos m\theta, \sin m\theta\}$). Hence for $f \in L^2(S)$, the decomposition promised by the theorem above takes the form

$$f = \sum_{m=-\infty}^{\infty} a_m e^{im\theta},$$

where the sum converges in $L^2(S)$. In other words, when $n = 2$ the decomposition given by the theorem above is just the standard Fourier series expansion of a function on the circle.

When $n > 2$, we can think of the theorem above as providing an expansion for functions $f \in L^2(S)$ analogous to the Fourier series expansion, with spherical harmonics playing the roles of the exponential functions $e^{im\theta}$ (or of the trigonometric functions $\cos m\theta, \sin m\theta$).

Inner Product of Spherical Harmonics

Suppose $p = \sum_\alpha b_\alpha x^\alpha$ and $q = \sum_\alpha c_\alpha x^\alpha$ are harmonic polynomials on $\mathbf{R}^n$. In this section we focus on the question of computing the inner product of p and q in $L^2(S)$. We denote this inner product by $\langle p, q\rangle$, although technically $\langle p|_S, q|_S\rangle$ would be more correct.

Each of p, q can be written as a sum of homogeneous harmonic polynomials, and we can expand the inner product $\langle p, q\rangle$ accordingly. By Proposition 5.9, the inner product of terms coming from the homogeneous parts of different degrees equals 0. In other words we could, if desired, assume that p and q are homogeneous harmonic polynomials of the same degree. Even then it appears that the best we could do would be to write

$$\langle p, q\rangle = \sum_\alpha \sum_\beta b_\alpha \overline{c_\beta} \int_S x^{\alpha+\beta}\, d\sigma(x).$$

The integral over S of the monomial $x^{\alpha+\beta}$ was explicitly calculated by Hermann Weyl in Section 3 of [20]; using that result would complete a formula for $\langle p, q\rangle$. We will take a different approach.

We have no right to expect the double-sum formula above to reduce to a single-sum formula of the form

$$\langle p, q\rangle = \sum_{\alpha} b_\alpha \overline{c_\alpha} w_\alpha,$$

because distinct mononials of the same degree are not necessarily orthogonal in $L^2(S)$. For example, ${x_1}^2$ and ${x_2}^2$ are not orthogonal in this space because their product is positive everywhere on S. However, a single-sum formula as above is the main result of this section; see Theorem 5.14.

The single-sum formula that we will prove makes it appear that the monomials form an orthonormal set in $L^2(S)$, which, as noted above, is not true. But we are dealing here only with harmonic polynomials, and no monomial of degree above 1 is harmonic. In some mysterious fashion being harmonic forces enough cancellation in the double sum to collapse it into a single sum.

The following lemma will be a key tool in our proof of the single-sum formula.

5.13 Lemma: *If $m > 0$ and $p, q \in \mathcal{H}_m(\mathbf{R}^n)$, then*

$$\int_S pq\, d\sigma = \frac{1}{m(n+2m-2)} \int_S \nabla p \cdot \nabla q\, d\sigma.$$

PROOF: Fix $m > 0$ and $p, q \in \mathcal{H}_m(\mathbf{R}^n)$. Using the homogeneity of pq, we see, just as in the proof of Proposition 5.9, that pq equals $(1/2m)$ times the normal derivative of pq on S. Thus

$$\begin{aligned}\int_S pq\, d\sigma &= \frac{1}{2mnV(B)} \int_S \nabla(pq) \cdot \mathbf{n}\, ds \\ &= \frac{1}{2mnV(B)} \int_B \Delta(pq)\, dV,\end{aligned}$$

where the $nV(B)$ term appears in the first equality because of the switch from normalized surface-area measure $d\sigma$ to surface-area measure ds (see A.2 in Appendix A) and the second equality comes from the divergence theorem (1.2). Convert the last integral to polar coordinates (1.5),

apply the product rule for the Laplacian (1.19), and use the homogeneity of ∇p and ∇q to get

$$\begin{aligned}\int_S pq\,d\sigma &= \frac{1}{m}\int_0^1 r^{n+2m-3}\int_S \nabla p\cdot\nabla q\,d\sigma\,dr\\ &= \frac{1}{m(n+2m-2)}\int_S \nabla p\cdot\nabla q\,d\sigma,\end{aligned}$$

as desired. ∎

Now we can prove the surprising single-sum formula for the inner product of two harmonic polynomials.

5.14 Theorem: *If $p = \sum_\alpha b_\alpha x^\alpha$ and $q = \sum_\alpha c_\alpha x^\alpha$ are harmonic polynomials on $\mathbf{R}^n$, then*

$$\langle p, q\rangle = \sum_\alpha b_\alpha \overline{c_\alpha} w_\alpha,$$

where

$$w_\alpha = \frac{\alpha!}{n(n+2)\dots(n+2|\alpha|-2)}.$$

PROOF: Every harmonic polynomial can be written as a finite sum of homogeneous harmonic polynomials. We already know (from Proposition 5.9) that homogeneous harmonic polynomials of different degrees are orthogonal in $L^2(S)$. Thus it suffices to prove the theorem under the assumption that $p, q \in \mathcal{H}_m(\mathbf{R}^n)$ for some nonnegative integer m. Because $\mathcal{H}_m(\mathbf{R}^n)$ is closed under complex conjugation, we can also assume, with no loss of generality, that each $c_\alpha \in \mathbf{R}$.

If $m = 0$, then p, q are constant and the desired result obviously holds (where the empty product in the denominator of the formula defining w_α is interpreted, as usual, to equal 1).

So fix $m > 0$ and assume, by induction, that the theorem holds for smaller values of m. Let $e_j = (0,\dots,0,1,0,\dots,0)$, where the 1 appears in the jth slot. Now $\nabla p\cdot\nabla q$ is a sum of terms, each of which is a product of harmonic polynomials. Thus using our induction hypothesis we have

$$\int_S \nabla p \cdot \nabla q \, d\sigma = \sum_{j=1}^{n} \int_S \Big(\sum_\alpha b_\alpha \alpha_j x^{\alpha - e_j}\Big)\Big(\sum_\alpha c_\alpha \alpha_j x^{\alpha - e_j}\Big) \, d\sigma(x)$$

$$= \sum_{j=1}^{n} \sum_\alpha b_\alpha c_\alpha \alpha_j{}^2 \frac{(\alpha - e_j)!}{n(n+2)\dots(n+2m-4)}$$

$$= \sum_\alpha b_\alpha c_\alpha \sum_{j=1}^{n} \alpha_j \frac{\alpha!}{n(n+2)\dots(n+2m-4)}$$

$$= \sum_\alpha b_\alpha c_\alpha \frac{\alpha! \, m}{n(n+2)\dots(n+2m-4)}.$$

The equation above, when combined with Lemma 5.13, gives the desired formula. ■

Spherical Harmonics Via Differentiation

Compute a few partial derivatives of the function $|x|^{2-n}$. You will find the answer is always of the same form—a polynomial divided by a power of $|x|$. For example,

$$D_1{}^2(|x|^{2-n}) = \frac{(2-n)(|x|^2 - nx_1{}^2)}{|x|^{n+2}}.$$

Notice here that the polynomial in the numerator is harmonic. This is no accident—differentiating $|x|^{2-n}$ exactly k times will always leave us with a homogeneous harmonic polynomial of degree k divided by $|x|^{n-2+2k}$, as we will see in Lemma 5.15. We will actually see much more than this, when we show (Theorem 5.18) that this procedure gives a formula for the canonical projection of $\mathcal{P}_m(\mathbf{R}^n)$ onto $\mathcal{H}_m(\mathbf{R}^n)$. This section concludes with the development of a fast algorithm for finding the Poission integral of a polynomial via differentiation (Theorem 5.21).

The Kelvin transform will play a key role here. To see why, observe that the Kelvin transform applied to the example in the paragraph above leaves us with the harmonic polynomial in the numerator. This indicates how we will obtain homogeneous harmonic polynomials—we first differentiate $|x|^{2-n}$, and then we apply the Kelvin transform.

For $p = \sum_\alpha c_\alpha x^\alpha$ a polynomial on $\mathbf{R}^n$, we define $p(D)$ to be the differential operator $\sum_\alpha c_\alpha D^\alpha$.

5.15 Lemma: *If $p \in \mathcal{P}_m(\mathbf{R}^n)$, then $K[p(D)|x|^{2-n}] \in \mathcal{H}_m(\mathbf{R}^n)$.*

PROOF: First we will show that $K[p(D)|x|^{2-n}] \in \mathcal{P}_m(\mathbf{R}^n)$. By linearity, we need only prove this in the special case when p is a mononomial. To get started, note that the desired result obviously holds when $m = 0$. Now we will use induction, assuming that the result holds for some fixed m, and then showing that it also holds for $m + 1$.

Let α be a multi-index with $|\alpha| = m$. By our induction hypothesis, there exists $u \in \mathcal{H}_m(\mathbf{R}^n)$ such that

$$K[D^\alpha|x|^{2-n}] = u.$$

Take the Kelvin transform of both sides of the equation above, getting

$$D^\alpha|x|^{2-n} = |x|^{2-n-2m}u.$$

Fix an index j, and differentiate both sides of the equation above with respect to x_j, getting

$$\begin{aligned} D_jD^\alpha|x|^{2-n} &= (2 - n - 2m)x_j|x|^{-n-2m}u + |x|^{2-n-2m}D_ju \\ &= |x|^{2-n-2(m+1)}[(2 - n - 2m)x_ju + |x|^2D_ju] \qquad \textbf{5.16} \\ &= |x|^{2-n-2(m+1)}v, \end{aligned}$$

where $v \in \mathcal{P}_{m+1}(\mathbf{R}^n)$. Now take the Kelvin transform of both sides of the equation above, getting

$$K[D_jD^\alpha|x|^{2-n}] = v.$$

Thus $K[D_jD^\alpha|x|^{2-n}] \in \mathcal{P}_{m+1}(\mathbf{R}^n)$. Because D_jD^α represents differentiation with respect to an arbitrary multi-index of order $m + 1$, this completes the induction argument.

All that remains is to prove that $K[p(D)|x|^{2-n}]$ is harmonic. But $|x|^{2-n}$ is harmonic and every partial derivative of any harmonic function is harmonic, so $p(D)|x|^{2-n}$ is harmonic. The proof is completed by recalling that the Kelvin transform of every harmonic function is harmonic (Theorem 4.7). ∎

Suppose $p \in \mathcal{P}_m(\mathbf{R}^n)$. Proposition 5.5 gives a unique decomposition

$$p = p_m + |x|^2 q,$$

where $p_m \in \mathcal{H}_m(\mathbf{R}^n)$ and $q \in \mathcal{P}_{m-2}(\mathbf{R}^n)$. The previous lemma states that $K[p(D)|x|^{2-n}] \in \mathcal{H}_m(\mathbf{R}^n)$. So p determines two harmonic polynomials, p_m and $K[p(D)|x|^{2-n}]$, leading to an investigation of the relationship between them. As we will see (Theorem 5.18), one of these harmonic polynomials is a constant multiple of the other, with the constant depending only on m and n. The key to proving this is the following lemma, which we can guess by looking at the proof of Lemma 5.15. Specifically, note that in 5.16, u gets multiplied by x_j, just as D^α is multiplied by D_j. An extra factor of $2 - n - m$ also appears. Thus 5.16 suggests the following lemma, where c_m is the constant defined by

$$c_m = \prod_{k=0}^{m-1} (2 - n - 2k).$$

Although c_m depends upon n as well as m, we are assuming that $n > 2$ is fixed. For $n = 2$, the definition of c_m and the analogue of the following lemma are given in Exercise 14 of this chapter.

5.17 Lemma: *If $n > 2$ and $p \in \mathcal{P}_m(\mathbf{R}^n)$, then*

$$K[p(D)|x|^{2-n}] = c_m(p - |x|^2 q)$$

for some $q \in \mathcal{P}_{m-2}(\mathbf{R}^n)$.

PROOF: The proof is a modification of the proof of the previous lemma. By linearity, we need only consider the case when p is a monomial. The desired result obviously holds when $m = 0$. Now we will use induction, assuming that the result holds for some fixed m, and then showing that it also holds for $m + 1$.

Let α be a multi-index with $|\alpha| = m$. By our induction hypothesis, there exists $q \in \mathcal{P}_{m-2}(\mathbf{R}^n)$ such that

$$K[D^\alpha|x|^{2-n}] = c_m(x^\alpha - |x|^2 q).$$

Follow the proof of Lemma 5.15, setting $u = c_m(x^\alpha - |x|^2 q)$, taking the Kelvin transform of both sides of the equation above, and then applying D_j to both sides, getting (see 5.16)

$$
\begin{aligned}
D_j D^\alpha |x|^{2-n} \\
&= |x|^{2-n-2(m+1)} c_m \Big[(2-n-2m) x_j (x^\alpha - |x|^2 q) + \frac{|x|^2 D_j u}{c_m} \Big] \\
&= |x|^{2-n-2(m+1)} c_{m+1} (x_j x^\alpha - |x|^2 v),
\end{aligned}
$$

where $v \in \mathcal{P}_{m-1}(\mathbf{R}^n)$. Now take the Kelvin transform of both sides of the equation above, getting

$$K[D_j D^\alpha |x|^{2-n}] = c_{m+1}(x_j x^\alpha - |x|^2 v).$$

Because $x_j x^\alpha$ represents an arbitrary monomial of order $m + 1$, this completes the induction argument and the proof. ∎

In the next theorem we will combine the last two lemmas. Recall that the canonical projection of $\mathcal{P}_m(\mathbf{R}^n)$ onto $\mathcal{H}_m(\mathbf{R}^n)$ comes from the decomposition given by Proposition 5.5. By the orthogonal projection onto $\mathcal{H}_m(S)$, we mean the usual orthogonal projection of the Hilbert space $L^2(S)$ onto the closed subspace $\mathcal{H}_m(S)$. In part (b) of the next theorem, to be formally correct we should have written $(p(D)|x|^{2-n})|_S/c_m$ instead of $p(D)|x|^{2-n}/c_m$.

5.18 Theorem: *Suppose $n > 2$ and $p \in \mathcal{P}_m(\mathbf{R}^n)$. Then:*

(a) *The canonical projection of p onto $\mathcal{H}_m(\mathbf{R}^n)$ is $K[p(D)|x|^{2-n}]/c_m$.*

(b) *The orthogonal projection of $p|_S$ onto $\mathcal{H}_m(S)$ is $p(D)|x|^{2-n}/c_m$.*

PROOF: By Lemma 5.17, we can write

5.19
$$p = K[p(D)|x|^{2-n}]/c_m + |x|^2 q$$

for some $q \in \mathcal{P}_{m-2}(\mathbf{R}^n)$. Lemma 5.15 shows that the first term on the right side of this equation is in $\mathcal{H}_m(\mathbf{R}^n)$. Thus this equation is the unique decomposition of p promised by Proposition 5.5, and furthermore $K[p(D)|x|^{2-n}]/c_m$ is the canonical projection of p onto $\mathcal{H}_m(\mathbf{R}^n)$, which proves (a).

To prove (b), restrict both sides of 5.19 to S, getting

$$p|_S = p(D)|x|^{2-n}/c_m + q|_S.$$

By Proposition 5.9, $q|_S$ is orthogonal to $\mathcal{H}_m(S)$. Thus taking the orthogonal projection onto $\mathcal{H}_m(S)$ of both sides of the equation above gives (b). ■

See Exercise 14 of this chapter for the analogue of the preceding theorem for $n = 2$.

As an immediate corollary of the theorem above, we get the following unusual identity for homogeneous harmonic polynomials.

5.20 Corollary: *If $n > 2$ and $p \in \mathcal{H}_m(\mathbf{R}^n)$, then*

$$p = K[p(D)|x|^{2-n}]/c_m.$$

Recall from Theorem 5.7 that for $p \in \mathcal{P}_m(\mathbf{R}^n)$, there is a unique decomposition of the form

$$p = p_m + |x|^2 p_{m-2} + \cdots + |x|^{2k} p_{m-2k},$$

where $k = [\frac{m}{2}]$ and each $p_j \in \mathcal{H}_j(\mathbf{R}^n)$. Recall also that the solution to the Dirichlet problem for B with boundary data $p|_S$ equals

$$p_m + p_{m-2} + \cdots + p_{m-2k}.$$

Part (a) of the previous theorem gives a fast algorithm for computing $p_m, p_{m-2}, \ldots, p_{m-2k}$ and thus for computing the Poisson integral of any polynomial. Specifically, p_m can be computed from the formula $p_m = K[p(D)|x|^{2-n}]/c_m$. Use this to then solve for $q \in \mathcal{P}_{m-2}(\mathbf{R}^n)$ in the decomposition $p = p_m + |x|^2 q$. To find p_{m-2}, repeat this procedure with q in place of p and $m - 2$ in place of m. Continue in this fashion, finding $p_m, p_{m-2}, \ldots, p_{m-2k}$.

The algorithm for computing the Poisson integral of a polynomial described in the paragraph above relies on differentiation rather than integration. We have found it typically to be several orders of magnitude faster than algorithms involving integration. The next theorem gives another algorithm, also using only differentiation, for the exact computation of Poisson integrals of polynomials. We have found it to be even faster than the algorithm described in the paragraph above, typically by a factor of about 2.

The algorithm described by the next theorem is used by the software discussed in Appendix B. This software shows, for example, that if $n = 5$ then the Poisson integral of ${x_1}^5 x_2$ equals

$$ {x_1}^5 x_2 - \frac{10{x_1}^3 x_2 |x|^2}{13} + \frac{15 x_1 x_2 |x|^4}{143} + \frac{10 {x_1}^3 x_2}{13} - \frac{10 x_1 x_2 |x|^2}{39} + \frac{5 x_1 x_2}{33}. $$

Note that in the solution above, the homogeneous part p_6 of highest order (the first three terms above) consists of the original function ${x_1}^5 x_2$ plus a polynomial multiple of $|x|^2$. This is expected, as we know that $p_6 = {x_1}^5 x_2 - |x|^2 q$ for some $q \in \mathcal{P}_4(\mathbf{R}^5)$.

Finally, we need one bit of notation. Define $\Delta^0 p = p$, and then for i a positive integer inductively define $\Delta^i p = \Delta(\Delta^{i-1} p)$.

In the theorem below, we could have obtained a formula for $c_{i,j}$ in closed form. However, in the inductive formulas given here are more efficient for computation. These formulas come from [4], which in turn partially based its derivation on ideas from [6].

5.21 Theorem: *Suppose $p \in \mathcal{P}_m(\mathbf{R}^n)$ has the decomposition*

$$ p = p_m + |x|^2 p_{m-2} + \cdots + |x|^{2k} p_{m-2k}, $$

where $k = [\frac{m}{2}]$ and each $p_{m-2j} \in \mathcal{H}_{m-2j}(\mathbf{R}^n)$. Then

$$ p_{m-2j} = \sum_{i=j}^{k} c_{i,j} |x|^{2(i-j)} \Delta^i p $$

for $j = 0, \dots, k$, where $c_{0,0} = 1$ and

$$ c_{j,j} = \frac{c_{j-1,j-1}(2m + n - 2j)}{2j(2m + n + 2 - 4j)(2m + n - 4j)} $$

for $j = 1, \dots, k$ and

$$ c_{i,j} = \frac{c_{i-1,j}}{2(j - i)(2m + n - 2 - 2j - 2i)} $$

for $i = j + 1, \dots, k$.

PROOF: As a special case of 4.5, we have

$$ \Delta(|x|^{2i} q) = 2i(2m + n - 2 - 2i)|x|^{2i-2} q $$

for $q \in \mathcal{H}_{m-2i}(\mathbf{R}^n)$. Repeated application of this equation shows that for every nonnegative integer j, the operator $|x|^{2j}\Delta^j$ equals a constant times the identity operator on $|x|^{2i}\mathcal{H}_{m-2i}(\mathbf{R}^n)$. Denoting this constant by $b_{i,j}$, note that $b_{i,j} = 0$ if and only if $j > i$. Furthermore, the reader should verify that

5.22 $$b_{j,j} = 2^j j! \prod_{i=1}^{j}(2m + n - 2j - 2i).$$

For $j = 0, \dots, k$, apply the operator $|x|^{2j}\Delta^j$ to both sides of the equation $p = \sum_{i=0}^{k} |x|^{2i} p_{m-2i}$, getting the lower-triangular system

5.23 $$|x|^{2j}\Delta^j p = \sum_{i=j}^{k} b_{i,j}|x|^{2i} p_{m-2i}.$$

Let $(c_{i,j})$ denote the matrix inverse of the $(k+1)$-by-$(k+1)$ matrix $(b_{i,j})$; thus $(c_{i,j})$ is also a lower-triangular matrix. View the system 5.23 as a matrix equation whose right side consists of the row matrix of unknowns $|x|^{2i}p_{m-2i}$ times the matrix $(b_{i,j})$. Now multiply (on the right) both sides of this matrix equation by the matrix $(c_{i,j})$ to solve for $|x|^{2i}p_{m-2i}$, then divide by $|x|^{2i}$ and interchange i and j to obtain

5.24 $$p_{m-2j} = \sum_{i=j}^{k} c_{i,j}|x|^{2(i-j)}\Delta^i p,$$

for $j = 0, \dots, k$, as desired.

The only remaining task is to prove the inductive formulas for $c_{j,j}$ and $c_{i,j}$. The diagonal entries of the inverse of a lower-triangular matrix are easy to compute. Specifically, we have $c_{j,j} = 1/b_{j,j}$. The claimed inductive formula for $c_{j,j}$ now follows from 5.22.

To prove the inductive formula for $c_{i,j}$, fix j and use 4.5 to take the Laplacian of both sides of 5.24, then multiply by $|x|^{2j}$, getting

$$\begin{aligned} 0 &= \sum_{i=j}^{k} c_{i,j}\Big(|x|^{2i}\Delta^{i+1}p + 2(i-j)(2m+n-2-2j-2i)|x|^{2i-2}\Delta^i p\Big) \\ &= \sum_{i=j+1}^{k} \Big(c_{i-1,j} + 2(i-j)(2m+n-2-2j-2i)c_{i,j}\Big)|x|^{2i-2}\Delta^i p, \end{aligned}$$

where the second equality is obtained from the first by breaking the sum into two parts, replacing i by $i-1$ in the first part, and recombining the two sums (after the change of summation, the first summation should go to $k+1$, but the $(k+1)$-term equals 0; similarly, the second sum should start at j, but the j-term equals 0).

The equality above must hold for all $p \in \mathcal{P}_m(\mathbf{R}^n)$ (the $c_{i,j}$ are independent of p). This can happen only if

$$c_{i-1,j} + 2(i-j)(2m+n-2-2j-2i)c_{i,j} = 0,$$

which gives our desired inductive formula. ∎

Explicit Bases of $\mathcal{H}_m(\mathbf{R}^n)$ and $\mathcal{H}_m(S)$

Theorem 5.18 implies that $\{K[D^\alpha|x|^{2-n}] : |\alpha| = m\}$ spans $\mathcal{H}_m(\mathbf{R}^n)$ and that $\{D^\alpha|x|^{2-n} : |\alpha| = m\}$ spans $\mathcal{H}_m(S)$. In the next theorem, we find an explicit subset of each of these spanning sets that is a basis.

5.25 Theorem: *If $n > 2$ then the set*

$$\{K[D^\alpha|x|^{2-n}] : |\alpha| = m \text{ and } \alpha_1 \le 1\}$$

is a vector space basis of $\mathcal{H}_m(\mathbf{R}^n)$, and the set

$$\{D^\alpha|x|^{2-n} : |\alpha| = m \text{ and } \alpha_1 \le 1\}$$

is a vector space basis of $\mathcal{H}_m(S)$.

PROOF: Let $\mathcal{B} = \{K[D^\alpha|x|^{2-n}] : |\alpha| = m \text{ and } \alpha_1 \le 1\}$. We will first show that $\mathcal{B}$ spans $\mathcal{H}_m(\mathbf{R}^n)$. For this we need only show that $K[D^\alpha|x|^{2-n}]$ is in the span of $\mathcal{B}$ for every multi-index α of degree m (by Theorem 5.18). So suppose α is a multi-index of degree m. If α_1 is 0 or 1, then $K[D^\alpha|x|^{2-n}]$ is in $\mathcal{B}$ by definition. Now we use induction on α_1. Suppose that $\alpha_1 > 1$ and that $K[D^\beta|x|^{2-n}]$ is in the span of $\mathcal{B}$ for all multi-indices β of degree m whose first components are less than α_1. Because $\Delta|x|^{2-n} \equiv 0$, we have

$$\begin{aligned} K[D^\alpha|x|^{2-n}] &= K[D_1{}^{\alpha_1-2}D_2{}^{\alpha_2}\dots D_n{}^{\alpha_n}(D_1{}^2|x|^{2-n})] \\ &= -K[D_1{}^{\alpha_1-2}D_2{}^{\alpha_2}\dots D_n{}^{\alpha_n}(\sum_{j=2}^{n} D_j{}^2|x|^{2-n})] \\ &= -\sum_{j=2}^{n} K[D_1{}^{\alpha_1-2}D_2{}^{\alpha_2}\dots D_n{}^{\alpha_n}(D_j{}^2|x|^{2-n})]. \end{aligned}$$

By our induction hypothesis, each of the summands in the last line is in the span of $\mathcal{B}$, and therefore $K[D^\alpha|x|^{2-n}]$ is in the span of $\mathcal{B}$. We conclude that $\mathcal{B}$ spans $\mathcal{H}_m(\mathbf{R}^n)$.

To complete the proof that $\mathcal{B}$ is a basis of $\mathcal{H}_m(\mathbf{R}^n)$, we show that the cardinality of $\mathcal{B}$ is at most the dimension of $\mathcal{H}_m(\mathbf{R}^n)$. We have $\mathcal{B} = \{K[D^\alpha|x|^{2-n}]\}$, where α ranges over multi-indices of length m that are not of the form $(\beta_1+2, \beta_2, \dots, \beta_n)$ with $|\beta| = m-2$. Therefore the cardinality of $\mathcal{B}$ is at most $\#\{\alpha : |\alpha| = m\} - \#\{\beta : |\beta| = m-2\}$, where # denotes cardinality. But from Proposition 5.5, we know that this difference equals the dimension of $\mathcal{H}_m(\mathbf{R}^n)$.

Having shown that $\mathcal{B}$ is a basis of $\mathcal{H}_m(\mathbf{R}^n)$, we can restrict to S, obtaining the second assertion of this theorem. ■

The software described in Appendix B uses Theorem 5.25 to construct bases of $\mathcal{H}_m(\mathbf{R}^n)$ and $\mathcal{H}_m(S)$. For example, this software produces the following vector space basis of $\mathcal{H}_4(\mathbf{R}^3)$:

$$\begin{aligned} &\{3|x|^4 - 30|x|^2x_2{}^2 + 35x_2{}^4, \\ &3|x|^2x_2x_3 - 7x_2{}^3x_3, \\ &|x|^4 - 5|x|^2x_2{}^2 - 5|x|^2x_3{}^2 + 35x_2{}^2x_3{}^2, \\ &3|x|^2x_2x_3 - 7x_2x_3{}^3, \\ &3|x|^4 - 30|x|^2x_3{}^2 + 35x_3{}^4, \\ &3|x|^2x_1x_2 - 7x_1x_2{}^3, \\ &|x|^2x_1x_3 - 7x_1x_2{}^2x_3, \\ &|x|^2x_1x_2 - 7x_1x_2x_3{}^2, \\ &3|x|^2x_1x_3 - 7x_1x_3{}^3\}. \end{aligned}$$

Although the previous theorem is valid only when $n > 2$, the $n = 2$ case is easy—earlier in the chapter we saw that $\{z^m, \overline{z^m}\}$ is a basis of $\mathcal{H}_m(\mathbf{R}^2)$ and $\{e^{im\theta}, e^{-im\theta}\}$ is a basis of $\mathcal{H}_m(S)$.

Zonal Harmonics

We continue to view $\mathcal{H}_m(S)$ as an inner product space with the $L^2(S)$-inner product. Fix a point $\eta \in S$, and consider the linear map $\Lambda\colon \mathcal{H}_m(S) \to \mathbf{C}$ defined by

$$\Lambda(p) = p(\eta).$$

Because $\mathcal{H}_m(S)$ is a finite-dimensional inner-product space, there exists a unique function $Z_m(\cdot, \eta) \in \mathcal{H}_m(S)$ such that

$$p(\eta) = \langle p, Z_m(\cdot, \eta)\rangle = \int_S p(\zeta)\overline{Z_m(\zeta, \eta)}\, d\sigma(\zeta)$$

for all $p \in \mathcal{H}_m(S)$. The spherical harmonic $Z_m(\cdot, \eta)$ is called the *zonal harmonic* of degree m with *pole* η. The terminology comes from geometric properties of Z_m that will be explained shortly.

We easily compute Z_m when $n = 2$. Clearly $Z_0 \equiv 1$. For $m > 0$, $\mathcal{H}_m(S)$ is the two-dimensional space spanned by $\{e^{im\theta}, e^{-im\theta}\}$, as we saw earlier. Thus if we fix $e^{i\varphi} \in S$, there are constants $\alpha, \beta \in \mathbf{C}$ such that $Z_m(e^{i\theta}, e^{i\varphi}) = \alpha e^{im\theta} + \beta e^{-im\theta}$. The reproducing property of the zonal harmonic then gives

$$\begin{aligned} \gamma e^{im\varphi} + \delta e^{-im\varphi} &= \int_0^{2\pi} (\gamma e^{im\theta} + \delta e^{-im\theta})(\overline{\alpha} e^{-im\theta} + \overline{\beta} e^{im\theta})\, \frac{d\theta}{2\pi} \\ &= \gamma\overline{\alpha} + \delta\overline{\beta} \end{aligned}$$

for every $\gamma, \delta \in \mathbf{C}$. Thus $\alpha = e^{-im\varphi}$ and $\beta = e^{im\varphi}$. We conclude that

5.26 $$Z_m(e^{i\theta}, e^{i\varphi}) = e^{im(\theta-\varphi)} + e^{im(\varphi-\theta)} = 2\cos m(\theta - \varphi).$$

Later (5.38) we will find an explicit formula for zonal harmonics in higher dimensions.

We now return to the case of arbitrary $n \geq 2$. The next proposition gives some basic properties of zonal harmonics. The proof of (c) below

uses orthogonal transformations, which play an important role in our study of zonal harmonics. We let $O(n)$ denote the group of orthogonal transformations on $\mathbf{R}^n$. Observe that $\mathcal{H}_m(\mathbf{R}^n)$ is $O(n)$-invariant, meaning that if $p \in \mathcal{H}_m(\mathbf{R}^n)$ and $T \in O(n)$, then $p \circ T \in \mathcal{H}_m(\mathbf{R}^n)$. It follows that $\mathcal{H}_m(S)$ is $O(n)$-invariant as well.

5.27 Proposition: *Suppose $\zeta, \eta \in S$ and $m \geq 0$. Then:*

(a) Z_m *is real valued.*

(b) $Z_m(\zeta, \eta) = Z_m(\eta, \zeta)$.

(c) $Z_m(\zeta, T(\eta)) = Z_m(T^{-1}(\zeta), \eta)$ *for all* $T \in O(n)$.

(d) $Z_m(\eta, \eta) = \dim \mathcal{H}_m(\mathbf{R}^n)$.

(e) $|Z_m(\zeta, \eta)| \leq \dim \mathcal{H}_m(\mathbf{R}^n)$.

PROOF: To prove (a), suppose $p \in \mathcal{H}_m(S)$ is real valued. Then

$$\begin{aligned} 0 &= \operatorname{Im} p(\eta) \\ &= \operatorname{Im} \int_S p(\zeta) \overline{Z_m(\zeta, \eta)}\, d\sigma(\zeta) \\ &= -\int_S p(\zeta) \operatorname{Im} Z_m(\zeta, \eta)\, d\sigma(\zeta). \end{aligned}$$

Defining p by $p(\zeta) = \operatorname{Im} Z_m(\zeta, \eta)$ yields

$$\int_S (\operatorname{Im} Z_m(\zeta, \eta))^2\, d\sigma(\zeta) = 0,$$

which implies $\operatorname{Im} Z_m \equiv 0$, proving (a).

To prove (b), consider any orthonormal basis $e_1, \dots, e_{h_m}$ of $\mathcal{H}_m(S)$, where $h_m = \dim \mathcal{H}_m(S) = \dim \mathcal{H}_m(\mathbf{R}^n)$ (see Proposition 5.8 for an explicit formula for h_m). By standard Hilbert space theory,

$$Z_m(\cdot, \eta) = \sum_{j=1}^{h_m} \langle Z_m(\cdot, \eta), e_j \rangle e_j = \sum_{j=1}^{h_m} \overline{e_j(\eta)} e_j.$$

Thus

5.28
$$Z_m(\zeta, \eta) = \sum_{j=1}^{h_m} \overline{e_j(\eta)} e_j(\zeta).$$

Because Z_m is real valued, the equation above is unchanged after complex conjugation, which implies (b).

To prove (c), let $T \in O(n)$. For every $p \in \mathcal{H}_m(S)$ we have

$$\begin{aligned} p(T(\eta)) &= (p \circ T)(\eta) \\ &= \int_S p(T(\zeta)) Z_m(\zeta, \eta)\, d\sigma(\zeta) \\ &= \int_S p(\zeta) Z_m(T^{-1}(\zeta), \eta)\, d\sigma(\zeta), \end{aligned}$$

the last equality following from the rotation invariance of σ. By the uniqueness of the zonal harmonic, the equation above gives (c).

To prove (d), note that taking $\zeta = T(\eta)$ in (c) gives

$$Z_m(T(\eta), T(\eta)) = Z_m(\eta, \eta).$$

Thus the function $\eta \mapsto Z_m(\eta, \eta)$ is constant on S. To evaluate this constant, take $\zeta = \eta$ in 5.28, obtaining

$$Z_m(\eta, \eta) = \sum_{j=1}^{h_m} |e_j(\eta)|^2.$$

Now integrate both sides of the equation above over S, getting

$$Z_m(\eta, \eta) = \int_S \Big(\sum_{j=1}^{h_m} |e_j(\eta)|^2 \Big)\, d\sigma(\eta) = h_m = \dim \mathcal{H}_m(\mathbf{R}^n),$$

which gives (d).

To prove (e), note that

$$\|Z_m(\cdot, \eta)\|_2^2 = \langle Z_m(\cdot, \eta), Z_m(\cdot, \eta) \rangle = Z_m(\eta, \eta) = \dim \mathcal{H}_m(\mathbf{R}^n),$$

where $\| \ \|_2$ denotes the norm in $L^2(S)$. Now

$$\begin{aligned} |Z_m(\zeta, \eta)| &= |\langle Z_m(\cdot, \zeta), Z_m(\cdot, \eta) \rangle| \\ &\le \|Z_m(\cdot, \zeta)\|_2 \|Z_m(\cdot, \eta)\|_2 \\ &= \dim \mathcal{H}_m(\mathbf{R}^n), \end{aligned}$$

completing the proof. ∎

Exercise 19 of this chapter deals with the question of when the inequality in part (e) of the proposition above is an equality.

Our previous decomposition $L^2(S) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m(S)$ has an elegant restatement in terms of zonal harmonics, as shown in the next theorem. Note that this is just the Fourier series decomposition when $n = 2$.

5.29 Theorem: *Suppose $f \in L^2(S)$. Let $p_m(\eta) = \langle f, Z_m(\cdot, \eta)\rangle$ for $m \geq 0$ and $\eta \in S$. Then $p_m \in \mathcal{H}_m(S)$ and*

$$f = \sum_{m=0}^{\infty} p_m$$

in $L^2(S)$.

PROOF: By Theorem 5.12, we can write $f = \sum_{m=0}^{\infty} q_m$ for some choice of $q_m \in \mathcal{H}_m(S)$, where the infinite sum converges in $L^2(S)$. The proof is completed by noticing that

$$p_m(\eta) = \langle f, Z_m(\cdot, \eta)\rangle = \langle \sum_{k=0}^{\infty} q_k, Z_m(\cdot, \eta)\rangle = \langle q_m, Z_m(\cdot, \eta)\rangle = q_m(\eta),$$

where the third equality comes from the orthogonality of spherical harmonics of different degrees (Proposition 5.9). ∎

The Poisson Kernel Revisited

Every element of $\mathcal{H}_m(S)$ has a unique extension to an element of $\mathcal{H}_m(\mathbf{R}^n)$; given $p \in \mathcal{H}_m(S)$, we will let p denote this extension as well. In particular, the notation $Z_m(\cdot, \zeta)$ will now often refer to the extension of this zonal harmonic to an element of $\mathcal{H}_m(\mathbf{R}^n)$.

Suppose $x \in \mathbf{R}^n$. If $x \neq 0$ and $p \in \mathcal{H}_m(\mathbf{R}^n)$, then

5.30
$$\begin{aligned} p(x) &= |x|^m p(x/|x|) \\ &= |x|^m \int_S p(\zeta) Z_m(x/|x|, \zeta)\, d\sigma(\zeta) \\ &= \int_S p(\zeta) Z_m(x, \zeta)\, d\sigma(\zeta). \end{aligned}$$

We easily check that the first and last terms above also agree when $x = 0$. Note that $Z_m(x, \cdot)$ is a spherical harmonic of degree m for each fixed $x \in \mathbf{R}^n$.

Our next result uses the equation above to expresses the Poisson integral of a polynomial in terms of zonal harmonics.

5.31 Proposition: *Let p be a polynomial on $\mathbf{R}^n$ of degree m. Then*

$$P[p|_S](x) = \sum_{k=0}^{m} \int_S p(\zeta) Z_k(x, \zeta)\, d\sigma(\zeta)$$

for every $x \in B$.

PROOF: By Theorem 5.1, $P[p|_S]$ is a polynomial of degree at most m and hence can be written in the form

5.32
$$P[p|_S] = \sum_{k=0}^{m} p_k,$$

where each $p_k \in \mathcal{H}_k(\mathbf{R}^n)$. For each $x \in B$ and each k we have

$$\begin{aligned} p_k(x) &= \int_S p_k(\zeta) Z_k(x, \zeta)\, d\sigma(\zeta) \\ &= \int_S \sum_{j=0}^{m} p_j(\zeta) Z_k(x, \zeta)\, d\sigma(\zeta) \\ &= \int_S p(\zeta) Z_k(x, \zeta)\, d\sigma(\zeta), \end{aligned}$$

where the first equality comes from 5.30, the second equality comes from the orthogonality of spherical harmonics of different degrees (see Proposition 5.9), and the third equality holds because p and its Poisson integral $\sum_{j=0}^{m} p_j$ agree on S. Combining the last equation with 5.32 gives the desired result. ∎

The proposition above leads us to the zonal harmonic expansion of the Poisson kernel.

5.33 Theorem: *For every $n \geq 2$,*

$$P(x,\zeta) = \sum_{m=0}^{\infty} Z_m(x,\zeta)$$

for all $x \in B$, $\zeta \in S$. The series converges absolutely and uniformly on $K \times S$ for every compact set $K \subset B$.

PROOF: For a fixed n, Proposition 5.27(e) and Exercise 10 of this chapter show that there exists a constant C such that

$$|Z_m(x,\zeta)| \leq Cm^{n-2}|x|^m$$

for all $x \in \mathbf{R}^n$, $\zeta \in S$. The series $\sum_{m=0}^{\infty} Z_m(x,\zeta)$ therefore has the desired convergence properties.

Fix $x \in B$. From Propositions 5.31 and 5.9 we see that

$$\int_S f(\zeta)P(x,\zeta)\,d\sigma(\zeta) = \int_S f(\zeta) \sum_{m=0}^{\infty} Z_m(x,\zeta)\,d\sigma(\zeta)$$

whenever f is the restriction of a polynomial to S. Because such functions are dense in $L^2(S)$, this implies that $P(x,\zeta) = \sum_{m=0}^{\infty} Z_m(x,\zeta)$ for almost every $\zeta \in S$. But all the functions involved are continuous, so we actually have equality everywhere, as desired. ■

When $n = 2$, we can express the theorem above in a familiar form. Recall that we used complex analysis (see 1.12) to show that the Poisson kernel for B_2 takes the form

$$P(re^{i\theta},e^{i\varphi}) = \sum_{m=-\infty}^{\infty} r^{|m|}e^{im(\theta-\varphi)} = 1 + \sum_{m=1}^{\infty} r^m 2\cos m(\theta-\varphi)$$

for all $r \in [0,1)$ and all $\theta, \varphi \in [0,2\pi]$. By 5.26, this is exactly the expansion in the theorem above.

The preceding theorem enables us to prove that the homogeneous expansion of an arbitrary harmonic function has the stronger convergence property discussed after Theorem 1.31.

5.34 Corollary: *If u is a harmonic function on $B(a,r)$, then there exist $p_m \in \mathcal{H}_m(\mathbf{R}^n)$ such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a)$$

for all $x \in B(a,r)$, the series converging absolutely and uniformly on compact subsets of $B(a,r)$.

PROOF: We first assume that u is harmonic on $\overline{B}$. For any $x \in B$, Theorem 5.33 gives

$$u(x) = \int_S u(\zeta)P(x,\zeta)\,d\sigma(\zeta) = \sum_{m=0}^{\infty} \int_S u(\zeta)Z_m(x,\zeta)\,d\sigma(\zeta).$$

Letting $p_m(x) = \int_S u(\zeta)Z_m(x,\zeta)\,d\sigma(\zeta)$ for $x \in \mathbf{R}^n$, observe that $p_m \in \mathcal{H}_m(\mathbf{R}^n)$. As in the proof of Theorem 5.33,

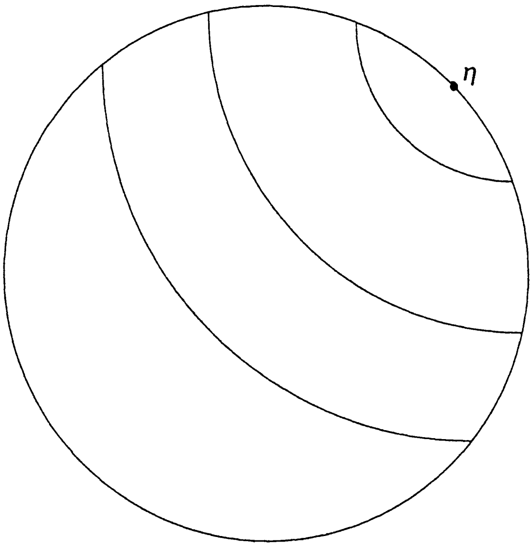
$$|p_m(x)| \le Cm^{n-2}|x|^m \int_S |u|\,d\sigma$$

for all $x \in \mathbf{R}^n$, and thus the series $\sum p_m$ converges absolutely and uniformly to u on compact subsets of B.

After a translation and dilation, the preceding argument shows that if u is harmonic on $B(a,r)$, then u has an expansion of the desired form in each $B(a,s)$, $0 < s < r$. By the uniqueness of homogeneous expansions, all of these expansions are the same, and thus u has the desired expansion on $B(a,r)$. ∎

A Geometric Characterization of Zonal Harmonics

In this section we give a simple geometric characterization of zonal harmonics. Recall the definition of a "parallel" from cartography: if we identify the surface of the earth with $S \subset \mathbf{R}^3$ so that the north pole is at $(0,0,1)$, then a parallel is simply the intersection of S with any plane perpendicular to the z-axis. The notion of a parallel is easily extended to all dimensions. Specifically, given $\eta \in S$, we define a *parallel orthogonal* to η to be the intersection of S with any hyperplane perpendicular to η.



Parallels orthogonal to η.

We claim that the zonal harmonic $Z_m(\cdot, \eta)$ is constant on each parallel orthogonal to η. To prove this, observe that a function f on S is constant on parallels orthogonal to η if and only if $f \circ T^{-1} = f$ for every $T \in O(n)$ with $T(\eta) = \eta$. Thus Proposition 5.27(c) proves our claim.

Our goal is this section is to show that scalar multiples of $Z_m(\cdot, \eta)$ are the only members of $\mathcal{H}_m(S)$ that are constant on parallels orthogonal to η (Theorem 5.37). This geometric property explains how zonal harmonics came to be named—the term "zonal" refers to the "zones" between parallels orthogonal to the "pole" η.

We will use two lemmas to prove our characterization of zonal harmonics. The first lemma describes the power series expansion of a real-analytic radial function.

5.35 Lemma: *If f is real analytic and radial on $\mathbf{R}^n$, then there exist constants $c_m \in \mathbf{C}$ such that*

$$f(x) = \sum_{m=0}^{\infty} c_m |x|^{2m}$$

for all x near 0.

PROOF: Assume first that $f \in \mathcal{P}_m(\mathbf{R}^n)$ and that f is not identically 0. Because f is radial, it has a constant value $c \neq 0$ on S, which implies that $f(x) = c|x|^m$ for all $x \in \mathbf{R}^n$. Clearly m is even (otherwise f would not be a polynomial). Thus f has the desired form in this case.

Now suppose that f is real analytic and radial, and that $\sum p_m$ is the homogeneous expansion of f near 0. Let $T \in O(n)$. Because f is radial, $f = f \circ T$, which gives $\sum p_m = \sum p_m \circ T$ near 0. Since p_m is a homogeneous polynomial of degree m, the same is true of $p_m \circ T$, so that $p_m = p_m \circ T$ for every m by the uniqueness of the homogeneous expansion of f. This is true for every $T \in O(n)$, and therefore each p_m is radial. The result in the previous paragraph now completes the proof. ∎

The next lemma is the final tool we need for our characterization of zonal harmonics. Recall that we can identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$, writing a typical point $z \in \mathbf{R}^n$ as $z = (x, y)$.

5.36 Lemma: *Suppose that u is harmonic on $\mathbf{R}^n$ and that $u(\cdot, y)$ is radial on $\mathbf{R}^{n-1}$ for each $y \in \mathbf{R}$. Suppose further that $u(0, y) = 0$ for all $y \in \mathbf{R}$. Then $u \equiv 0$.*

PROOF: Recall that the power series of a function harmonic on $\mathbf{R}^n$ converges everywhere on $\mathbf{R}^n$ (see Exercise 34 in Chapter 1). Because u is real analytic on $\mathbf{R}^n$ and each $u(\cdot, y)$ is radial on $\mathbf{R}^{n-1}$, Lemma 5.35 implies that the expansion of u takes the form

$$u(x, y) = \sum_{m=0}^{\infty} c_m(y)|x|^{2m},$$

where each c_m is a real-analytic function of y. Because u is harmonic, we obtain

$$\begin{aligned} 0 &= \Delta u(x, y) \\ &= \sum_{m=0}^{\infty} c_m''(y)|x|^{2m} + \sum_{m=1}^{\infty} \alpha_m c_m(y)|x|^{2(m-1)} \\ &= \sum_{m=0}^{\infty} [c_m''(y) + \alpha_{m+1} c_{m+1}(y)]|x|^{2m}, \end{aligned}$$

where $\alpha_m = 2m(2m + n - 3)$. Looking at the last series, we see that each term in brackets vanishes. Because $c_0(y) = u(0, y) = 0$, we easily verify by induction that each c_m is identically zero. Thus $u \equiv 0$, as desired. ∎

Let **N** denote the north pole $(0, \ldots, 0, 1)$. We can now characterize the zonal harmonics geometrically.

5.37 Theorem: *Let $\eta \in S$. A spherical harmonic of degree m is constant on parallels orthogonal to η if and only if it is a constant multiple of $Z_m(\cdot, \eta)$.*

PROOF: We have already seen that $Z_m(\cdot, \eta)$ is constant on parallels orthogonal to η.

For the converse, we may assume $m \geq 1$. For convenience we first treat the case $\eta = \mathbf{N}$. So suppose $p \in \mathcal{H}_m(\mathbf{R}^n)$ is constant on parallels orthogonal to $\mathbf{N}$. For $T \in O(n-1)$, we then have

$$p(Tx, y) = p(x, y)$$

for all $(x, y) \in S$, and hence for all $(x, y) \in \mathbf{R}^n$. Because this holds for all $T \in O(n-1)$, we conclude that $p(\cdot, y)$ is radial on $\mathbf{R}^{n-1}$ for each $y \in \mathbf{R}$. In particular, $Z_m((\cdot, y), \mathbf{N})$, regarded as an element of $\mathcal{H}_m(\mathbf{R}^n)$, is radial on $\mathbf{R}^{n-1}$ for each $y \in \mathbf{R}$.

Now choose c such that $p(\mathbf{N}) = cZ_m(\mathbf{N}, \mathbf{N})$, and define

$$u = p - cZ_m(\cdot, \mathbf{N}).$$

Then u is harmonic on $\mathbf{R}^n$, $u(\cdot, y)$ is radial on $\mathbf{R}^{n-1}$ for each $y \in \mathbf{R}$, and $u(0, y) = u(y\mathbf{N}) = y^m u(\mathbf{N}) = 0$ for every $y \in \mathbf{R}$. By Lemma 5.36, $u \equiv 0$. Thus p is a constant multiple of $Z_m(\cdot, \mathbf{N})$, as desired.

For the general $\eta \in S$, choose $T \in O(n)$ such that $T(\mathbf{N}) = \eta$. If $p \in \mathcal{H}_m(S)$ is constant on parallels orthogonal to η, then $p \circ T$ is constant on parallels orthogonal to $\mathbf{N}$. Hence $p \circ T$ is a constant multiple of $Z_m(\cdot, \mathbf{N})$, which implies that p is a constant multiple of $Z_m(\cdot, \mathbf{N}) \circ T^{-1}$, which, by Proposition 5.27(c), equals $Z_m(\cdot, \eta)$. ∎

An Explicit Formula for Zonal Harmonics

The expansion of the Poisson kernel given by Theorem 5.33 allows us to find an explicit formula for the zonal harmonics.

5.38 Theorem: *Let $x \in \mathbf{R}^n$ and let $\zeta \in S$. Then $Z_m(x, \zeta)$ equals*

$$(n+2m-2) \sum_{k=0}^{[m/2]} (-1)^k \frac{n(n+2)\dots(n+2m-2k-4)}{2^k k!\,(m-2k)!} (x \cdot \zeta)^{m-2k} |x|^{2k}$$

for each $m > 0$.

PROOF: The function $(1-z)^{-n/2}$ is holomorphic on the unit disk in the complex plane, and so it has a power series expansion

5.39
$$(1-z)^{-n/2} = \sum_{k=0}^{\infty} c_k z^k$$

for $|z| < 1$. We easily compute that

5.40
$$c_k = \frac{(\frac{n}{2})(\frac{n}{2}+1)\dots(\frac{n}{2}+k-1)}{k!}.$$

Fix $\zeta \in S$. For $|x|$ small, 5.39 and the binomial formula imply that

$$\begin{aligned} P(x,\zeta) &= (1-|x|^2)(1+|x|^2-2x\cdot\zeta)^{-n/2} \\ &= (1-|x|^2) \sum_{k=0}^{\infty} c_k (2x\cdot\zeta - |x|^2)^k \\ &= (1-|x|^2) \sum_{k=0}^{\infty} c_k \sum_{j=0}^{k} (-1)^j \binom{k}{j} 2^{k-j} (x\cdot\zeta)^{k-j} |x|^{2j}. \end{aligned}$$

By Theorem 5.33, $Z_m(\cdot,\zeta)$ is equal to the sum of the terms of degree m in the power series representation of $P(\cdot,\zeta)$. Thus the formula above implies that

5.41
$$Z_m(x,\zeta) = q_m(x) - |x|^2 q_{m-2}(x),$$

where q_m and q_{m-2} are the sums of terms of degree m and $m-2$, respectively, in the double series above. It is easy to see that

$$q_m(x) = \sum_{m/2 \le k \le m} c_k(-1)^{m-k}\binom{k}{m-k}2^{2k-m}(x\cdot\zeta)^{2k-m}|x|^{2m-2k}.$$

Replacing the index k by $m-k$ in this sum shows that

$$q_m(x) = \sum_{k=0}^{[m/2]} c_{m-k}(-1)^k\binom{m-k}{k}2^{m-2k}(x\cdot\zeta)^{m-2k}|x|^{2k}.$$

Using 5.40, the last equation becomes

$$q_m(x) = \sum_{k=0}^{[m/2]} (-1)^k \frac{n(n+2)\dots(n+2m-2k-2)}{2^k k!\,(m-2k)!}(x\cdot\zeta)^{m-2k}|x|^{2k}.$$

By replacing m by $m-2$, we obtain a formula for q_{m-2}. In that formula, replace the index k by $k-1$, and then combine terms in 5.41 to complete the proof. ∎

Note that for $x \in S$, the expansion in the theorem above shows that $Z_m(x,\zeta)$ is a function of $x\cdot\zeta$. We could have predicted this by recalling from the last section that on S, the zonal harmonic $Z_m(\cdot,\zeta)$ is constant on parallels orthogonal to ζ.

The formula for zonal harmonics given by the theorem above may be combined with Proposition 5.31 and the formula for the integral over S of any monomomial ([20], Section 3) to calculate explicitly the Poisson integral of any polynomial. However, this procedure is typically several orders of magnitude slower than the algorithm given by Theorem 5.21.

Exercises

1. Suppose p is a polynomial on $\mathbf{R}^n$ such that $p|_S \equiv 0$. Prove that there exists a polynomial q such that $p = (1 - |x|^2)q$.

2. Suppose p is a homogeneous polynomial on $\mathbf{R}^n$ and $u = P[p|_S]$. Prove·that u is a homogeneous polynomial with the same degree as p if and only if p is harmonic.

3. Suppose p is a polynomial on $\mathbf{R}^n$ and $u = P[p|_S]$. Prove that the degree of u is less than the degree of p if and only if the homogenenous part of p of highest degree is a polynomial multiple of $|x|^2$.

4. Suppose that f is a homogeneous polynomial on $\mathbf{R}^n$ of even (respectively, odd) degree. Prove that $P[f]$ is a polynomial consisting only of terms of even (respectively, odd) degree.

5. Suppose E is an open ellipsoid in $\mathbf{R}^n$.

   (a) Prove that if p is a polynomial on $\mathbf{R}^n$ of degree at most m, then there exists a harmonic polynomial q on $\mathbf{R}^n$ of degree at most m such that $q|_{\partial E} = p|_{\partial E}$.

   (b) Use part (a) and the Stone-Weierstrass Theorem to show that if $f \in C(E)$, then there exists $u \in C(\overline{E})$ such that $u|_{\partial E} = f$ and u is harmonic on E.

6. Let f be a polynomial on $\mathbf{R}^n$. Prove that $P_e[f|_S]$, the exterior Poisson integral of $f|_S$ (see Chapter 4), extends to a function that is harmonic on $\mathbf{R}^n \setminus \{0\}$.

7. *Generalized Dirichlet Problem:* Show that if f and g are polynomials on $\mathbf{R}^n$, then there is a unique polynomial p with $p|_S = f|_S$ and $\Delta p = g$. (The software described in Appendix B can find p explicitly.)

8. From Pascal's triangle we know $\binom{N+1}{M} = \binom{N}{M} + \binom{N}{M-1}$. Use this and Proposition 5.8 to show that

$$\dim \mathcal{H}_m(\mathbf{R}^n) = \binom{n+m-2}{n-2} + \binom{n+m-3}{n-2}$$

for $m \geq 1$.

9. Prove that $\dim \mathcal{H}_m(\mathbf{R}^n) < \dim \mathcal{H}_{m+1}(\mathbf{R}^n)$ when $n > 2$.

10. Prove that for a fixed n,

$$\frac{\dim \mathcal{H}_m(\mathbf{R}^n)}{m^{n-2}} \to \frac{2}{(n-2)!}$$

as $m \to \infty$.

11. Prove that

$$\int_S |x_1{}^2 - x_2{}^2|^2 \, d\sigma(x) = \frac{4}{n(n+2)}.$$

12. Suppose $p, q \in \mathcal{H}_m(\mathbf{R}^n)$. Prove that

$$p(D)[q] = n(n+2)\dots(n+2m-2)\int_S pq \, d\sigma.$$

(Note that the left side of the equation above, which appears to be a function, is actually a constant because p and q are both homogeneous polynomials of degree m.)

13. Where in the proof of Lemma 5.17 was the hypothesis $n > 2$ used?

14. For $n = 2$, let $c_m = (-2)^{m-1}(m-1)!$. Suppose $m > 0$ and $p \in \mathcal{P}_m(\mathbf{R}^2)$.

    (a) Prove that

$$K[p(D)\log|x|] = c_m(p - |x|^2 q)$$

    for some $q \in \mathcal{P}_{m-2}(\mathbf{R}^2)$.

    (b) Prove that the orthogonal projection of p onto $\mathcal{H}_m(\mathbf{R}^2)$ is $K[p(D)\log|x|]/c_m$.

15. Given a polynomial f on $\mathbf{R}^n$, how would you go about determining whether or not $f|_S$ is a spherical harmonic?

16. Prove that if $p \in \mathcal{H}_m(\mathbf{R}^n)$, then

$$D_j K[p] = K[|x|^2 D_j p + (2 - n - 2m)x_j p]$$

for $1 \le j \le n$.

17. Prove that if $p \in \mathcal{H}_m(\mathbf{R}^n)$, where $n > 2$ and $m > 0$, then

$$K[p] = \frac{1}{m(4-n-2m)} \sum_{j=1}^{n} D_j K[D_j p].$$

18. Let $f \in C(S)$. The *Neumann problem* for B with boundary data f is to find a function harmonic on $\overline{B}$ whose outward normal derivative on S equals f and whose value at the origin is 0.

    (a) Show that the Neumann problem with boundary data f has at most one solution.

    (b) Show if the Neumann problem with boundary data f has a solution, then $\int_S f \, d\sigma = 0$.

    (c) Show that if p is a polynomial on $\mathbf{R}^n$, then the Neumann problem with boundary data $p|_S$ has a solution if and only if $\int_S p \, d\sigma = 0$. Describe how you would calculate a solution to the Neumann problem with boundary data $p|_S$ from the solution to the Dirichlet problem with boundary data $p|_S$, and vice versa.

19. (a) For $n = 2$, find a necessary and sufficient condition for equality in Proposition 5.27(e).

    (b) Prove that if $n > 2$ and $m > 0$, then the inequality in Proposition 5.27(e) is an equality if and only if $\zeta = \eta$ or $\zeta = -\eta$.

20. Define $P_M(x, \zeta) = \sum_{m=0}^{M} Z_m(x, \zeta)$. Show that for fixed $\zeta \in S$,

$$\inf_{x \in B} P_M(x, \zeta) \to -\infty \quad \text{as } M \to \infty,$$

even though for each fixed $x \in B$, $P_M(x, \zeta) \to P(x, \zeta) > 0$ as $M \to \infty$ (by Theorem 5.33).

21. Fix $x \in B$. For $f = P(x, \cdot)$, what is the expansion given by Theorem 5.29? Show how this could be used to give an alternative proof of Theorem 5.33.

22. Give an example of a real-analytic function on B whose homogeneous expansion (about 0) does not converge in all of B. (Compare this with Corollary 5.34.)

23. Suppose $u \in C^1(B)$ is such that the function $x \mapsto x \cdot \nabla u(x)$ is harmonic on B. Prove that u is harmonic on B.

24. Suppose u is harmonic on $\mathbf{R}^n$ and u is constant on parallels orthogonal to $\eta \in S$. Show that there exist $c_0, c_1, \ldots \in \mathbf{C}$ such that

$$u(x) = \sum_{m=0}^{\infty} c_m Z_m(x, \eta)$$

for all $x \in \mathbf{R}^n$.

25. Suppose that u is harmonic on $\overline{B}$, u is constant on parallels orthogonal to $\eta \in S$, and $u(r\eta) = 0$ for infinitely many $r \in [-1, 1]$. Prove that $u \equiv 0$ on B.

26. Show that u need not vanish identically in Exercise 25 if "harmonic on $\overline{B}$" is replaced by "continuous on $\overline{B}$ and harmonic on B". (*Suggestion:* Set $q_m(x) = Z_m(x, \eta)/(\dim \mathcal{H}_m(\mathbf{R}^n))$ and consider a sum of the form $\sum_{k=1}^{\infty}(-1)^k c_k q_{m_k}(x)$, where the coefficients c_k are positive and summable and the integers m_k are widely spaced.)

27. Show that there exists a nonconstant harmonic function u on $\mathbf{R}^2$ that is constant on parallels orthogonal to $e^{i\theta}$ as well as on parallels orthogonal to $e^{i\varphi}$ if and only if $\theta - \varphi$ is a rational multiple of π.

28. Suppose $n > 2$ and $\zeta, \eta \in S$. Under what conditions can a function on S be constant on parallels orthogonal to ζ as well as on parallels orthogonal to η?

29. Fix a positive integer m. By Theorem 5.38, there is a polynomial q of one variable such that $Z_m(\eta, \zeta) = q(\eta \cdot \zeta)$ for all $\eta, \zeta \in S$. Prove that if n is even, then each coefficient of q is an integer.

CHAPTER 6

Harmonic Hardy Spaces

Poisson Integrals of Measures

In Chapter 1 we defined the Poisson integral of a function $f \in C(S)$ to be the function $P[f]$ on B given by

6.1 $$P[f](x) = \int_S f(\zeta)P(x,\zeta)\,d\sigma(\zeta).$$

We now extend this definition: for μ a complex Borel measure on S, the Poisson integral of μ, denoted $P[\mu]$, is the function on B defined by

6.2 $$P[\mu](x) = \int_S P(x,\zeta)\,d\mu(\zeta).$$

Differentiating under the integral sign in 6.2, we see that $P[\mu]$ is harmonic on B.

The set of complex Borel measures on S will be denoted by $M(S)$. The total variation norm of $\mu \in M(S)$ will be denoted by $\|\mu\|$. Recall that $M(S)$ is a Banach space under the total variation norm. By the Riesz Representation Theorem, if we identify $\mu \in M(S)$ with the linear functional Λ_μ on $C(S)$ given by

$$\Lambda_\mu(f) = \int_S f\,d\mu,$$

then $M(S)$ is isometrically isomorphic to the dual space of $C(S)$. (A good source for these results is [15].)

We will also deal with the Banach spaces $L^p(S), 1 \le p \le \infty$. When $p \in [1, \infty)$, $L^p(S)$ consists of the Borel measurable functions f on S for which

$$\|f\|_p = \left(\int_S |f|^p \, d\sigma\right)^{1/p} < \infty;$$

$L^\infty(S)$ consists of the Borel measurable functions f on S for which $\|f\|_\infty < \infty$, where $\|f\|_\infty$ denotes the essential supremum norm on S with respect to σ. The number $q \in [1, \infty]$ is said to be *conjugate* to p if $1/p + 1/q = 1$. If $1 \le p < \infty$ and q is conjugate to p, then $L^q(S)$ is the dual space of $L^p(S)$. Here we identify $g \in L^q(S)$ with the linear functional Λ_g on $L^p(S)$ defined by

$$\Lambda_g(f) = \int_S fg \, d\sigma.$$

Note that because σ is a finite measure on S, $L^p(S) \subset L^1(S)$ for all $p \in [1, \infty]$. Recall also that $C(S)$ is dense in $L^p(S)$ for $1 \le p < \infty$.

It is natural to identify each $f \in L^1(S)$ with the measure $\mu_f \in M(S)$ defined on Borel sets $E \subset S$ by

6.3
$$\mu_f(E) = \int_E f \, d\sigma.$$

Shorthand for 6.3 is the expression $d\mu_f = f \, d\sigma$. The map $f \mapsto \mu_f$ is a linear isometry of $L^1(S)$ into $M(S)$. We will often identify functions in $L^1(S)$ as measures in this manner without further comment.

For $f \in L^1(S)$, we will write $P[f]$ in place of $P[\mu_f]$. Here one could also try to define $P[f]$ as in 6.1. Fortunately the two definitions agree, because if φ is a bounded Borel measurable function on S (in particular, if $\varphi = P(x, \cdot)$), then $\int_S \varphi \, d\mu_f = \int_S \varphi f \, d\sigma$. Our notation is thus consistent with that defined previously for continuous functions on S.

Throughout this chapter, when given a function u on B, the notation u_r will refer to the function on S defined by $u_r(\zeta) = u(r\zeta)$; here, of course, $0 \le r < 1$.

We know that if $f \in C(S)$, then $P[f]$ has a continuous extension to $\overline{B}$. What can be said of the more general Poisson integrals defined above? We begin to answer this question in the next two theorems.

6.4 Theorem: *The following growth estimates apply to Poisson integrals:*

(a) *If* $\mu \in M(S)$ *and* $u = P[\mu]$, *then* $\|u_r\|_1 \le \|\mu\|$ *for all* $r \in [0, 1)$.

(b) *If* $1 \le p \le \infty$, $f \in L^p(S)$, *and* $u = P[f]$, *then* $\|u_r\|_p \le \|f\|_p$ *for all* $r \in [0, 1)$.

PROOF: The identity

6.5
$$P(r\eta, \zeta) = P(r\zeta, \eta),$$

valid for all $\eta, \zeta \in S$ and all $r \in [0, 1)$, will be used to prove both (a) and (b).

To prove (a), let $\mu \in M(S)$ and set $u = P[\mu]$. For $\eta \in S$ and $r \in [0, 1)$,

$$|u(r\eta)| \le \int_S P(r\eta, \zeta)\, d|\mu|(\zeta),$$

where $|\mu|$ denotes the total variation measure associated with μ. Fubini's theorem and 6.5 then give

$$\begin{aligned} \|u_r\|_1 &= \int_S |u(r\eta)|\, d\sigma(\eta) \\ &\le \int_S \int_S P(r\eta, \zeta)\, d|\mu|(\zeta)\, d\sigma(\eta) \\ &= \int_S \int_S P(r\zeta, \eta)\, d\sigma(\eta)\, d|\mu|(\zeta) \\ &= \|\mu\|. \end{aligned}$$

For (b), assume first that $1 \le p < \infty$. Let $f \in L^p(S)$ and set $u = P[f]$. Then

$$|u(r\eta)| \le \int_S |f(\zeta)|\, P(r\eta, \zeta)\, d\sigma(\zeta).$$

By Jensen's inequality,

$$|u(r\eta)|^p \le \int_S |f(\zeta)|^p P(r\eta, \zeta)\, d\sigma(\zeta).$$

Integrate this last expression over S and use an argument similar to that given for (a) to get $\|u_r\|_p \le \|f\|_p$, as desired.

The case $f \in L^\infty(S)$ is the easiest. With $u = P[f]$, we have

$$\begin{aligned}|u(r\eta)| &\le \int_S |f(\zeta)|\, P(r\eta,\zeta)\, d\sigma(\zeta)\\ &\le \|f\|_\infty \int_S P(r\eta,\zeta)\, d\sigma(\zeta)\\ &= \|f\|_\infty.\end{aligned}$$

■

Our first consequence of the last theorem is that $\|u_r\|_p$ is an increasing function of r for each harmonic function u. A necessary and sufficient condition for the inequality in the corollary below to be an equality is given in Exercise 4 of this chapter.

6.6 Corollary: *If u is harmonic on B and $0 \le r \le s < 1$, then*

$$\|u_r\|_p \le \|u_s\|_p$$

for all $p \in [1, \infty]$.

PROOF: Suppose u is harmonic on B and $0 \le r \le s < 1$. The idea of the proof is to think of u_r as a dilate of the Poisson integral of u_s; then the result follows from the previous theorem. More specifically,

$$\|u_r\|_p = \|P[u_s]_{\frac{r}{s}}\|_p \le \|u_s\|_p,$$

where the equality follows from Theorem 1.21 and the inequality follows from Theorem 6.4(b). ■

If $f \in C(S)$ and $u = P[f]$, we know that $u_r \to f$ in $C(S)$ as $r \to 1$. This fact and Theorem 6.4 enable us to prove the following result on L^p-convergence.

6.7 Theorem: *Suppose $1 \le p < \infty$. If $f \in L^p(S)$ and $u = P[f]$, then $\|u_r - f\|_p \to 0$ as $r \to 1$.*

PROOF: Let $p \in [1, \infty)$, let $f \in L^p(S)$, and set $u = P[f]$. Fix $\varepsilon > 0$, and choose $g \in C(S)$ with $\|f - g\|_p < \varepsilon$. Setting $v = P[g]$, we have

$$\|u_r - f\|_p \le \|u_r - v_r\|_p + \|v_r - g\|_p + \|g - f\|_p.$$

Now $(u_r - v_r) = (P[f - g])_r$, hence $\|u_r - v_r\|_p < \varepsilon$ by Theorem 6.4. Note also that $\|v_r - g\|_p \leq \|v_r - g\|_\infty$. Thus

$$\|u_r - f\|_p < \|v_r - g\|_\infty + 2\varepsilon.$$

Because $g \in C(S)$, we have $\|v_r - g\|_\infty \to 0$ as $r \to 1$. It follows that $\limsup_{r\to 1} \|u_r - f\|_p \leq 2\varepsilon$. Since ε is arbitrary, $\|u_r - f\|_p \to 0$, as desired. ∎

Theorem 6.7 fails when $p = \infty$. In fact, for $f \in L^\infty(S)$ and $u = P[f]$, we have $\|u_r - f\|_\infty \to 0$ as $r \to 1$ if and only if $f \in C(S)$, as the reader should verify.

In the case $\mu \in M(S)$ and $u = P[\mu]$, one might ask if the L^1-functions u_r always converge to μ in $M(S)$. Here as well the answer is negative. Because $L^1(S)$ is a closed subspace of $M(S)$, $u_r \to \mu$ in $M(S)$ precisely when μ is absolutely continuous with respect to σ.

We will see in the next section that there is a weak sense in which convergence at the boundary occurs in the cases discussed in the two paragraphs above.

Weak* Convergence

A useful concept in analysis is the notion of weak* convergence. Suppose X is a normed linear space and X^* is the dual space of X. If $\Lambda_1, \Lambda_2, \ldots \in X^*$, then the sequence (Λ_k) is said to converge *weak** to $\Lambda \in X^*$ provided $\lim_{k\to\infty} \Lambda_k(x) = \Lambda(x)$ for every $x \in X$. In other words, $\Lambda_k \to \Lambda$ weak* precisely when the sequence (Λ_k) converges pointwise on X to Λ. We will also deal with one-parameter families $\{\Lambda_r : r \in [0,1)\} \subset X^*$; here we say that $\Lambda_r \to \Lambda$ weak* if $\Lambda_r(x) \to \Lambda(x)$ as $r \to 1$ for each fixed $x \in X$.

A simple observation we need later is that if $\Lambda_k \to \Lambda$ weak*, then

6.8 $$\|\Lambda\| \leq \liminf_{k\to\infty} \|\Lambda_k\|.$$

Here $\|\Lambda\|$ is the usual operator norm on the dual space X^* defined by $\|\Lambda\| = \sup\{|\Lambda(x)| : x \in X, \|x\| \leq 1\}$.

Convergence in norm implies weak* convergence, but the converse is false. A simple example is furnished by ℓ^2, the space of square

summable sequences. Because ℓ^2 is a Hilbert space, $(\ell^2)^* = \ell^2$. Let e_k denote the element of ℓ^2 that has 1 in the kth slot and 0's elsewhere. Then for each $a = (a_1, a_2, \dots) \in \ell^2$, $\langle a, e_k \rangle = a_k$ for every k. Thus $e_k \to 0$ weak* in ℓ^2 as $k \to \infty$, while $\|e_k\| = 1$ in the ℓ^2-norm for every k. This example also shows that inequality may occur in 6.8.

Our next result is the replacement for Theorem 6.7 in the cases mentioned at the end of the last section.

6.9 Theorem: *Poisson integrals have the following weak* convergence properties:*

(a) *If $\mu \in M(S)$ and $u = P[\mu]$, then $u_r \to \mu$ weak* in $M(S)$ as $r \to 1$.*

(b) *If $f \in L^\infty(S)$ and $u = P[f]$, then $u_r \to f$ weak* in $L^\infty(S)$ as $r \to 1$.*

PROOF: Recall that $C(S)^* = M(S)$. Suppose $\mu \in M(S)$, $u = P[\mu]$, and $g \in C(S)$. To prove (a), we need to show that

6.10
$$\int_S g u_r \, d\sigma \to \int_S g \, d\mu$$

as $r \to 1$.

Working with the left-hand side of 6.10, we have

$$\begin{aligned} \int_S g u_r \, d\sigma &= \int_S g(\zeta) \int_S P(r\zeta, \eta) \, d\mu(\eta) \, d\sigma(\zeta) \\ &= \int_S \int_S g(\zeta) P(r\eta, \zeta) \, d\sigma(\zeta) \, d\mu(\eta) \\ &= \int_S P[g](r\eta) \, d\mu(\eta), \end{aligned}$$

where we have used 6.5 again. Because $g \in C(S)$, $P[g](r\eta) \to g(\eta)$ uniformly on S as $r \to 1$. This proves 6.10 and completes the proof of (a).

The proof of (b) is similar. We first recall that $L^1(S)^* = L^\infty(S)$. With $f \in L^\infty(S)$ and $u = P[f]$, we thus need to show that

6.11
$$\int_S g u_r \, d\sigma. \to \int_S g f \, d\sigma$$

as $r \to 1$, for each $g \in L^1(S)$. Using the same manipulations as above, we see that the left side of 6.11 equals $\int_S (P[g])_r f \, d\sigma$. By Theorem 6.7, $(P[g])_r \to g$ in $L^1(S)$ as $r \to 1$. Because $f \in L^\infty(S)$, we have $(P[g])_r f \to gf$ in $L^1(S)$. This proves 6.11, completing the proof of (b). ∎

In Chapter 2 we told the reader that every bounded harmonic function on B is the Poisson integral of a bounded measurable function on S. In Chapter 3, we claimed that each positive harmonic function on B is the Poisson integral of some positive measure on S. We will prove these results in the next section. The key to these proofs is the following fundamental theorem on weak* convergence.

6.12 Theorem: *If X is a separable normed linear space, then every norm-bounded sequence in X^* contains a weak* convergent subsequence.*

PROOF: Assume (Λ_m) is a norm-bounded sequence in X^*. Then (Λ_m) is both pointwise bounded and equicontinuous on X (equicontinuity follows from the linearity of the functionals Λ_m). By the Arzela-Ascoli Theorem for separable metric spaces (Theorem 11.28 in [15]), (Λ_m) contains a subsequence (Λ_{m_k}) converging uniformly on compact subsets of X. In particular, (Λ_{m_k}) converges pointwise on X, which implies that (Λ_{m_k}) converges weak* to some element of X^*. ∎

In the next section we will apply the preceding theorem to the separable Banach spaces $C(S)$ and $L^q(S), 1 \le q < \infty$.

The Spaces $h^p(B)$

The estimates obtained in Theorem 6.4 suggest the definition of some new function spaces. For $1 \le p \le \infty$, we define $h^p(B)$ to be the class of functions u harmonic on B for which

$$\|u\|_{h^p} = \sup_{0 \le r < 1} \|u_r\|_p < \infty.$$

Thus $h^p(B)$ consists of the harmonic functions on B whose L^p-norms on spheres centered at the origin are uniformly bounded. Because

$\|u_r\|_p$ is an increasing function of r for each harmonic function u (Corollary 6.6), we have

$$\|u\|_{h^p} = \lim_{r \to 1} \|u_r\|_p$$

for each $u \in h^p(B)$. Note that $h^\infty(B)$ is the collection of bounded harmonic functions on B, and that

$$\|u\|_{h^\infty} = \sup_{x \in B} |u(x)|.$$

We refer to the spaces $h^p(B)$ as "harmonic Hardy spaces". The usual "Hardy spaces", denoted by $H^p(B_2)$, consist of the functions in $h^p(B_2)$ that are holomorphic on B_2; they are named in honor of the mathematician G. H. Hardy, who first studied them.

It is straightforward to verify that $h^p(B)$ is a normed linear space under the norm $\| \ \|_{h^p}$. A consequence of Theorem 6.13 below is that $h^p(B)$ is a Banach space.

Here are some observations that can be elegantly stated in terms of the h^p-spaces:

(a) The map $\mu \to P[\mu]$ is a linear isometry of $M(S)$ into $h^1(B)$.

(b) For $1 < p \le \infty$, the map $f \to P[f]$ is a linear isometry of $L^p(S)$ into $h^p(B)$.

Let us verify these claims. First, the maps in question are clearly linear. Second, in the case of (a), we have

$$\|P[\mu]\|_{h^1} \le \|\mu\|$$

by Theorem 6.4. On the other hand, 6.8 and Theorem 6.9 show that

$$\|\mu\| \le \liminf_{r \to 1} \|(P[\mu])_r\|_1 = \|P[\mu]\|_{h^1}.$$

This proves (a). The proof of (b) when $p = \infty$ is similar. The proof of (b) when $1 < p < \infty$ is even easier, following from Theorem 6.7.

We now prove the remarkable result that the maps in (a) and (b) above are onto.

6.13 Theorem: *The Poisson integral induces the following surjective isometries:*

(a) *The map $\mu \mapsto P[\mu]$ is a linear isometry of $M(S)$ onto $h^1(B)$.*

(b) *For $1 < p \le \infty$, the map $f \mapsto P[f]$ is a linear isometry of $L^p(S)$ onto $h^p(B)$.*

PROOF: All that remains to be verified in (a) is that the range of the map $\mu \mapsto P[\mu]$ is all of $h^1(B)$. To prove this, suppose $u \in h^1(B)$. By definition, this means that the family $\{u_r : r \in [0,1)\}$ is norm-bounded in $L^1(S)$, and hence in $M(S) = C(S)^*$. Theorem 6.12 thus implies there exists a sequence $r_j \to 1$ such that the sequence u_{r_j} converges weak* to some $\mu \in M(S)$. The proof of (a) will be completed by showing that $u = P[\mu]$.

Fix $x \in B$. Because the functions $y \mapsto u(r_j y)$ are harmonic on $\overline{B}$, we have

6.14 $$u(r_j x) = \int_S u(r_j \zeta) P(x, \zeta)\, d\sigma(\zeta)$$

for each j. Now let $j \to \infty$. Simply by continuity, the left side of 6.14 converges to $u(x)$. On the other hand, because $P(x,\cdot) \in C(S)$, the right side of 6.14 converges to $P[\mu](x)$. Therefore $u(x) = P[\mu](x)$, and thus $u = P[\mu]$ on B, as desired.

The proof of (b) is similar. Fix $p \in (1, \infty]$, let $u \in h^p(B)$, and let q be the number conjugate to p. Then the family $\{u_r : r \in [0,1)\}$ is norm-bounded in $L^p(S) = L^q(S)^*$. By Theorem 6.12, there exists a sequence $r_j \to 1$ such that u_{r_j} converges weak* to some $f \in L^p(S)$. The argument given in the paragraph above may now be used, essentially verbatim, to show that $u = P[f]$; the difference is that here we need to observe that $P(x,\cdot) \in L^q(S)$. We leave it to the reader to fill in the rest of the proof. ■

The theorem above contains the assertion made in Chapter 2 that if u is bounded and harmonic on B, then $u = P[f]$ for some $f \in L^\infty(S)$. We next take up the claim made in Chapter 3.

6.15 Corollary: *If u is positive and harmonic on B, then there is a unique positive measure $\mu \in M(S)$ such that $u = P[\mu]$.*

PROOF: Suppose u is positive and harmonic on B. Then

$$\int_S |u_r|\,d\sigma = \int_S u_r\,d\sigma = u(0)$$

for every $r \in [0,1)$, the last equality following from the mean-value property. Thus $u \in h^1(B)$, which by Theorem 6.13 means that there is a unique $\mu \in M(S)$ such that $u = P[\mu]$. Being the weak* limit of the positive measures u_r (Theorem 6.9(a)), μ is itself positive. ■

Our next proposition gives a growth estimate for functions in $h^p(B)$. For a slight improvement of this proposition, see Exercise 11 of this chapter.

6.16 Proposition: *Suppose* $1 \le p < \infty$. *If* $u \in h^p(B)$, *then*

$$|u(x)| \le \left(\frac{1+|x|}{(1-|x|)^{n-1}}\right)^{1/p} \|u\|_{h^p}$$

for all $x \in B$.

PROOF: Suppose $u \in h^p(B)$ and $x \in B$. First consider the case where $1 < p < \infty$. By Theorem 6.13, there exists $f \in L^p(S)$ such that $u = P[f]$; furthermore $\|u\|_{h^p} = \|f\|_p$. Let q be the number conjugate to p. Now

$$|u(x)| = \left|\int_S f(\zeta)P(x,\zeta)\,d\sigma(\zeta)\right|$$

6.17
$$\le \left(\int_S P(x,\zeta)^q\,d\sigma(\zeta)\right)^{1/q}\|u\|_{h^p}.$$

Notice that

$$\int_S P(x,\zeta)^q\,d\sigma(\zeta) \le \sup_{\zeta\in S} P(x,\zeta)^{q-1}\int_S P(x,\zeta)\,d\sigma(\zeta)$$

6.18
$$= \left(\frac{1+|x|}{(1-|x|)^{n-1}}\right)^{q-1}$$

Combining 6.17 and 6.18 gives the desired result.

The $p = 1$ case is similar and is left to the reader. ■

We conclude this section with a result that will be useful in the next chapter.

6.19 Theorem: *Let $\zeta \in S$. Suppose that u is positive and harmonic on B, and that u extends continuously to $\overline{B} \setminus \{\zeta\}$ with $u = 0$ on $S \setminus \{\zeta\}$. Then there exists a positive constant c such that*

$$u = cP(\cdot, \zeta).$$

PROOF: We have $u = P[\mu]$ for some positive $\mu \in M(S)$ by Theorem 6.15, and we have $u_r \to \mu$ weak* in $M(S)$ as $r \to 1$ by Theorem 6.9. The hypotheses on u imply that the functions u_r converge to 0 uniformly on compact subsets of $S \setminus \{\zeta\}$ as $r \to 1$. Therefore $\int_S \varphi \, d\mu = 0$ for any continuous φ on S that is zero near ζ. This implies that $\mu(S \setminus \{\zeta\}) = 0$, and thus that μ is a point mass at ζ. The conclusion of the theorem is immediate from this last statement. ■

The Hilbert Space $h^2(B)$

The map $f \mapsto P[f]$ is a linear isometry of $L^2(S)$ onto $h^2(B)$ (by Theorem 6.13). Because $L^2(S)$ is a Hilbert space, we can use this isometry to transfer a Hilbert space structure to $h^2(B)$. Specifically, we can define

$$\langle P[f], P[g] \rangle = \langle f, g \rangle = \int_S f \overline{g} \, d\sigma$$

for $f, g \in L^2(S)$, where we use $\langle \ , \ \rangle$ to denote the inner product on both $h^2(B)$ and $L^2(S)$, allowing the context to make clear which is intended.

Given $u, v \in h^2(B)$, it would be nice to have an intrinsic formula for $\langle u, v \rangle$ that does not involve finding $f, g \in L^2(S)$ such that $u = P[f]$ and $v = P[g]$. Fortunately, Theorem 6.7 leads to such a formula. We have $u_r \to f$ and $v_r \to g$ in $L^2(S)$, and thus $\langle u_r, v_r \rangle \to \langle f, g \rangle$. Hence

$$\langle u, v \rangle = \lim_{r \to 1} \int_S u(r\zeta) \overline{v(r\zeta)} \, d\sigma(\zeta).$$

For $f \in L^2(S)$ and $x \in B$, we have

6.20 $$P[f](x) = \langle f, P(x, \cdot) \rangle.$$

To translate this to an intrinsic formula on $h^2(B)$, we need to find the Poisson integral of $P(x,\cdot)$. In other words, we need to extend $P(x,\cdot)$, which is currently defined on S, to a harmonic function on B. To do this, note that

$$P(x,\zeta) = \frac{1-|x|^2}{|x-\zeta|^n} = \frac{1-|x|^2}{(1-2x\cdot\zeta+|x|^2)^{n/2}}$$

for $\zeta \in S$. We extend the domain of P by defining

6.21
$$P(x,y) = \frac{1-|x|^2|y|^2}{(1-2x\cdot y+|x|^2|y|^2)^{n/2}}$$

for all $x, y \in \mathbf{R}^n \times \mathbf{R}^n$ for which the denominator above is not 0. Note that this agrees with our previous definition when $y \in S$.

Our extended Poisson kernel P has the pleasant properties that $P(x,y) = P(y,x)$ and $P(x,y) = P(|x|y, x/|x|)$. The last equation shows that for x fixed, $P(x,\cdot)$ is a harmonic function (because it is a dilate of a harmonic function). In particular, for $x \in B$, the function $P(x,\cdot)$ is harmonic on $\overline{B}$ and hence is the function in $h^2(B)$ that corresponds to the unextended Poisson kernel $P(x,\cdot) \in L^2(S)$. The extended Poisson kernel will play a major role when we study Bergman spaces in Chapter 8.

Translating 6.20 to $h^2(B)$, we have the intrinsic formula

6.22
$$u(x) = \langle u, P(x,\cdot)\rangle$$

for all $x \in B$ and $u \in h^2(B)$. The usefulness of this viewpoint is demonstrated by the next proposition, which gives a sharp growth estimate for functions in $h^2(B)$, slightly better than the $p = 2$ case of Proposition 6.16.

6.23 Proposition: *If $u \in h^2(B)$, then*

$$|u(x)| \le \sqrt{\frac{1+|x|^2}{(1-|x|^2)^{n-1}}}\,\|u\|_{h^2}$$

for all $x \in B$.

PROOF: Suppose $u \in h^2(B)$ and $x \in B$. From the Cauchy-Schwarz inequality and 6.22, we have

$$|u(x)| \leq \|P(x,\cdot)\|_{h^2}\|u\|_{h^2}.$$

Now

$$\|P(x,\cdot)\|_{h^2}^2 = \langle P(x,\cdot), P(x,\cdot)\rangle = P(x,x),$$

where the last equality comes from 6.22. Use 6.21 to compute $P(x,x)$ and complete the proof. ∎

The Schwarz Lemma

The Schwarz Lemma in complex analysis states that if h is holomorphic on B_2 with $|h| < 1$ and $h(0) = 0$, then $|h(z)| \leq |z|$ for all $z \in B_2$; furthermore, if equality holds at any nonzero $z \in B_2$, then $h(z) = \lambda z$ for all $z \in B_2$, where λ is a complex number of modulus one. In this section we take up the Schwarz Lemma for functions harmonic on B_n.



Hermann Amandus Schwarz (1843-1921), whose reflection principle we used in Chapter 4 and whose lemma we now extend to harmonic functions, is also noted for his discovery of a procedure for solving the Dirichlet problem.

Let S^+ denote the northern hemisphere $\{\zeta \in S : \zeta_n > 0\}$ and let S^- denote the southern hemisphere $\{\zeta \in S : \zeta_n < 0\}$. We define a harmonic function $U = U_n$ on B by setting

$$U = P[\chi_{S^+} - \chi_{S^-}].$$

In other words, U is the Poisson integral of the function that equals 1 on S^+ and -1 on S^-. Note that U is harmonic on B, with $|U| < 1$ and $U(0) = 0$.

The following theorem shows that U and its rotations are the extremal functions for the Schwarz Lemma for harmonic functions. Recall that $\mathbf{N} = (0, \dots, 0, 1)$ denotes the north pole of S.

6.24 Harmonic Schwarz Lemma: *Suppose that u is harmonic on B, $|u| < 1$ on B, and $u(0) = 0$. Then*

$$|u(x)| \le U(|x|\mathbf{N})$$

for every $x \in B$. Equality holds for some nonzero $x \in B$ if and only if $u = \lambda(U \circ T)$, where λ is a complex constant of modulus 1 and T is an orthogonal transformation.

PROOF: Fix $x \in B$. After a rotation, we can assume that x lies on the radius from 0 to $\mathbf{N}$, so that $x = |x|\mathbf{N}$.

First we consider the case where u is real valued. By Theorem 6.13, there is a real-valued function $f \in L^\infty(S)$ such that $u = P[f]$ and $\|f\|_\infty \le 1$.

We claim that $u(x) \le U(x)$. This inequality is equivalent to the inequality

$$\int_{S^-} (1 + f(\zeta))P(x, \zeta)\, d\sigma \le \int_{S^+} (1 - f(\zeta))P(x, \zeta)\, d\sigma.$$

Because $x = |x|\mathbf{N}$, we have $P(x, \zeta) = (1 - |x|^2)/(1 + |x|^2 - 2|x|\zeta_n)^{n/2}$, so the inequality above is equivalent to

$$\textbf{6.25} \qquad \int_{S^-} \frac{1 + f(\zeta)}{(1 + |x|^2 - 2|x|\zeta_n)^{n/2}}\, d\sigma(\zeta) \le \int_{S^+} \frac{1 - f(\zeta)}{(1 + |x|^2 - 2|x|\zeta_n)^{n/2}}\, d\sigma(\zeta).$$

The condition $u(0) = 0$ implies that $\int_{S^-} f\, d\sigma = -\int_{S^+} f\, d\sigma$. Thus, since ζ_n is negative on S^- and positive on S^+, we have

$$\begin{aligned}\int_{S^-} \frac{1+f(\zeta)}{(1+|x|^2-2|x|\zeta_n)^{n/2}}\,d\sigma(\zeta) &\le \int_{S^-} \frac{1+f(\zeta)}{(1+|x|^2)^{n/2}}\,d\sigma(\zeta)\\ &= \int_{S^+} \frac{1-f(\zeta)}{(1+|x|^2)^{n/2}}\,d\sigma(\zeta)\\ &\le \int_{S^+} \frac{1-f(\zeta)}{(1+|x|^2-2|x|\zeta_n)^{n/2}}\,d\sigma(\zeta).\end{aligned}$$

Thus 6.25 holds, completing the proof that $u(x) \le U(x)$. Note that if $x \ne 0$, then the last two inequalities are equalities if and only if $f = 1$ almost everywhere on S^+ and $f = -1$ almost everywhere on S^-. In other words, we have $u(x) = U(x)$ if and only if $u = U$.

Now remove the restriction that u be real valued. Choose $\beta \in \mathbf{C}$ such that $|\beta| = 1$ and $\beta u(x) = |u(x)|$. Apply the result just proved to the real part of βu, getting $|u(x)| \le U(x)$, with equality if and only if $\beta u = U$. ∎

Note that while the extremal functions for the Schwarz Lemma for holomorphic functions are the entire functions $z \mapsto \lambda z$ (with $|\lambda| = 1$), the extremal functions for the Harmonic Schwarz Lemma are discontinuous at the boundary of B. Later in this section we will give a concrete formula for U when $n = 2$; Exercise 24 of this chapter gives formulas for $U(|x|\mathbf{N})$ when $n = 3, 4$. The software package described in Appendix B can compute $U(|x|\mathbf{N})$ for higher values of n.

The Schwarz Lemma for holomorphic functions has a second part that we did not mention earlier. Specifically, if h is holomorphic on B_2 and $|h| < 1$ on B_2, then $|h'(0)| \le 1$; equality holds if and only if $h(z) = \lambda z$ for some constant λ of modulus one. (Almost all complex analysis texts add the hypothesis that $h(0) = 0$, which is not needed for this part of the Schwarz Lemma.) The next theorem gives the corresponding result for harmonic functions. Here the gradient takes the place of the holomorphic derivative.

6.26 Theorem: *Suppose u is a real-valued harmonic function on B_n and $|u| < 1$ on B_n. Then*

$$|(\nabla u)(0)| \le \frac{2V(B_{n-1})}{V(B_n)}.$$

Equality holds if and only if $u = U \circ T$ for some orthogonal transformation T.

PROOF: We begin by investigating the size of the partial derivative $D_n u(0)$. By Theorem 6.13, there is a real-valued function $f \in L^\infty(S)$ such that $u = P[f]$ and $\|f\|_\infty \le 1$. Differentiating under the integral sign in the Poisson integral formula, we have

$$\begin{aligned} D_n u(0) &= \int_S f(\zeta) D_n P(0,\zeta)\, d\sigma(\zeta) \\ &= n \int_S f(\zeta)\zeta_n\, d\sigma(\zeta) \\ &\le n \int_S |\zeta_n|\, d\sigma(\zeta). \end{aligned}$$

Equality holds here if and only if $f = 1$ almost everywhere on S^+ and $f = -1$ almost everywhere on S^-, which is equivalent to saying that u equals U. The last integral can be easily evaluated using A.6 from Appendix A:

$$\begin{aligned} n \int_S |\zeta_n|\, d\sigma(\zeta) &= \frac{2}{V(B_n)} \int_{B_{n-1}} (1 - |x|^2)^{-1/2}(1 - |x|^2)^{1/2}\, dV_{n-1}(x) \\ &= \frac{2V(B_{n-1})}{V(B_n)}. \end{aligned}$$

Thus $D_n u(0) \le 2V(B_{n-1})/V(B_n)$, with equality if and only if $u = U$.

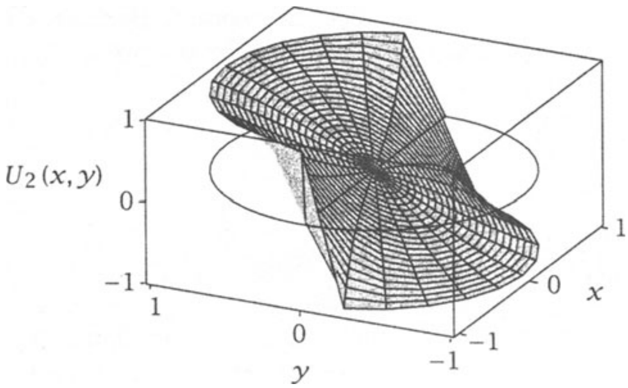
Applying this result to rotations of u, we see that every directional derivative of u at 0 is bounded above by $2V(B_{n-1})/V(B_n)$; the length of $\nabla u(0)$ is therefore bounded by the same constant, with equality if and only if u is a rotation of U. ∎

The bound given above on $|(\nabla u)(0)|$ could not be improved if we added the hypothesis that $u(0) = 0$, because the extremal function already satisfies that condition.

When $n = 2$, the preceding theorem shows that $|(\nabla u)(0)| \le 4/\pi$. Note that the optimal constant $4/\pi$ is larger than 1, which is the optimal constant for the Schwarz Lemma for holomorphic functions.

Theorem 6.26 fails for complex-valued harmonic functions (Exercise 23 of this chapter). The gradient, which points in the direction of maximal increase for a real-valued function, seems to have no natural geometric interpretation for complex-valued functions.

We now derive an explicit formula for the extremal function U when $n = 2$. Here the arctangent of a real number is always taken to lie in the interval $(-\pi/2, \pi/2)$.



The graph of the harmonic function U_2 along with the boundary of its domain. On the upper-half of unit circle in the xy-plane, this function equals 1; on the lower half of the circle it equals -1.

6.27 Proposition: *Let $z = (x, y)$ be a point in B_2. Then*

$$U_2(x,y) = \frac{2}{\pi}\arctan\frac{2y}{1-x^2-y^2}$$

and

$$U_2(|z|\mathbf{N}) = \frac{4}{\pi}\arctan|z|.$$

PROOF: Think of $z = x + iy$ as a complex variable. The conformal map $z \mapsto (1+z)/(1-z)$ takes B_2 onto the right half-plane. The function $z \mapsto \log[(1+z)/(1-z)]$, where log denotes the principal branch of the logarithm, is therefore holomorphic on B_2. Multiplying the imaginary part of this function by $2/\pi$, we see that the function u defined by

$$u(x,y) = \frac{2}{\pi}\arctan\frac{2y}{1-x^2-y^2}$$

is harmonic on B_2.

Because u is bounded on B_2, Theorem 6.13 implies that $u = P[f]$ on B_2 for some $f \in L^\infty(S)$. Theorem 6.9 shows that $u_r \to f$ weak* in $L^\infty(S)$ as $r \to 1$. But note that u extends to be continuous on $B_2 \cup S^+ \cup S^-$, with $u = 1$ on S^+ and $u = -1$ on S^-; thus $u_r \to u|_S$ weak* in $L^\infty(S)$ as $r \to 1$ (by the dominated convergence theorem). Hence $f = u|_S$ almost everywhere on S. Thus $u = U_2$, completing the proof of the first part of the proposition.

The second assertion in the proposition now follows from standard double-angle identities from trigonometry. ∎

The Fatou Theorem

Recall the cones $\Gamma_\alpha(a)$ defined in the section *Limits Along Rays* of Chapter 2. We will use these cones to define nontangential approach regions in the ball. For $\alpha > 0$ and $\zeta \in S$, we first translate and rotate $\Gamma_\alpha(0)$ to obtain a new cone with vertex ζ and axis of symmetry containing the origin. This new cone crashes through the sphere on the side opposite of ζ, making it unsuitable for a nontangential approach region in B. To fix this, consider the largest ball $B(0, r(\alpha))$ contained in the new cone (we do not need to know the exact value of $r(\alpha)$). Taking the convex hull of $B(0, r(\alpha))$ and the point ζ, and then removing the point ζ, we obtain the open set $\Omega_\alpha(\zeta)$ pictured here.

The region $\Omega_\alpha(\zeta)$ has the properties we seek for a nontangential approach region in B with vertex ζ. Specifically, $\Omega_\alpha(\zeta)$ stays away from the sphere except near ζ, and near ζ it equals the translated and rotated version of $\Gamma_\alpha(0)$ with which we started.

We have $\Omega_\alpha(\zeta) \subset \Omega_\beta(\zeta)$ if $\alpha < \beta$, and B is the union of the sets $\Omega_\alpha(\zeta)$ as α ranges over $(0, \infty)$.

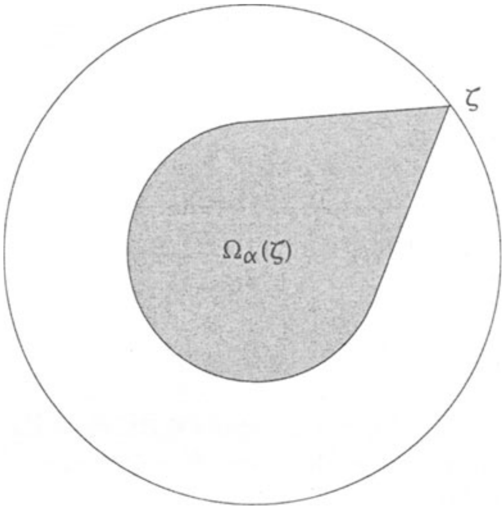
Note that

$$T(\Omega_\alpha(\zeta)) = \Omega_\alpha(T(\zeta))$$

for every orthogonal transformation T on $\mathbf{R}^n$. This allows us to transfer statements about the geometry of, say, $\Omega_\alpha(\mathbf{N})$ to any $\Omega_\alpha(\zeta)$.

A function u on B is said to have *nontangential limit* L at $\zeta \in S$ if for each $\alpha > 0$, we have $u(x) \to L$ as $x \to \zeta$ within $\Omega_\alpha(\zeta)$.

In this section we prove that if $u \in h^1(B)$, then u has a nontangential limit at almost every $\zeta \in S$. (In this chapter, the term "almost everywhere" will mean "almost everywhere with respect to σ".) Theorems



The nontangential approach region $\Omega_\alpha(\zeta)$.

asserting the almost everywhere existence of limits within approach regions are commonly referred to as "Fatou Theorems". The first such result was proved by Fatou [8], who in 1906 showed that bounded harmonic functions in the open unit disk have nontangential limits almost everywhere on the unit circle.

We approach the Fatou theorem for $h^1(B)$ via several operators known as "maximal functions". Given a function u on B and $\alpha > 0$, the *nontangential maximal function* of u, denoted by $\mathcal{N}_\alpha[u]$, is the function on S defined by

$$\mathcal{N}_\alpha[u](\zeta) = \sup_{x \in \Omega_\alpha(\zeta)} |u(x)|.$$

The *radial maximal function* of u, denoted by $\mathcal{R}[u]$, is the function on S defined by

$$\mathcal{R}[u](\zeta) = \sup_{0 \le r < 1} |u(r\zeta)|.$$

Clearly $\mathcal{R}[u](\zeta) \le \mathcal{N}_\alpha[u](\zeta)$ for every $\zeta \in S$ and every $\alpha > 0$. The following theorem shows that, up to a constant multiple, the reverse inequality holds for positive harmonic functions on B.

6.28 Theorem: *For every $\alpha > 0$, there exists a constant $C_\alpha < \infty$ such that*

$$\mathcal{N}_\alpha[u](\zeta) \le C_\alpha \mathcal{R}[u](\zeta)$$

for all $\zeta \in S$ and all positive harmonic functions u on B.

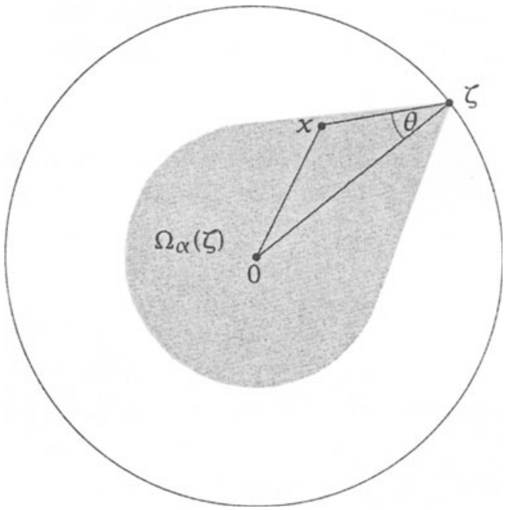
PROOF: Let $\zeta \in S$. The theorem then follows immediately from the existence of a constant C_α such that

6.29 $$P(x, \eta) \le C_\alpha P(|x|\zeta, \eta)$$

for all $x \in \Omega_\alpha(\zeta)$ and all $\eta \in S$. To prove 6.29, apply the law of cosines to the triangle with vertices 0, x, and ζ in 6.30 to see that there is a constant A_α such that

$$|x - \zeta| < A_\alpha(1 - |x|)$$

for all $x \in \Omega_\alpha(\zeta)$.



6.30 *$|x - \zeta|$ is comparable to $(1 - |x|)$ for $x \in \Omega_\alpha(\zeta)$.*

Thus

$$\big||x|\zeta - \eta\big| \le \big||x|\zeta - x\big| + |x - \eta|$$
$$\le |\zeta - x| + |x - \eta|$$
$$\le (A_\alpha + 1)|x - \eta|$$

for all $x \in \Omega_\alpha(\zeta)$ and all $\eta \in S$. This shows that 6.29 holds with $C_\alpha = (1 + A_\alpha)^n$. ■

We turn now to a key operator in analysis, the Hardy-Littlewood maximal function. For $\zeta \in S$ and $\delta > 0$, define

$$\kappa(\zeta, \delta) = \{\eta \in S : |\eta - \zeta| < \delta\}.$$

Thus $\kappa(\zeta, \delta)$ is the open "spherical cap" on S with center ζ and radius δ. (Note that $\kappa(\zeta, \delta) = S$ when $\delta > 2$.) The *Hardy-Littlewood maximal function* of $\mu \in M(S)$, denoted by $\mathcal{M}[\mu]$, is the function on S defined by

$$\mathcal{M}[\mu](\zeta) = \sup_{\delta > 0} \frac{|\mu|(\kappa(\zeta, \delta))}{\sigma(\kappa(\zeta, \delta))}.$$

Suppose $\mu \in M(S)$ is positive and $\delta > 0$ is fixed. Let (ζ_j) be a sequence in S such that $\zeta_j \to \zeta$. Because the characteristic functions of $\kappa(\zeta_j, \delta)$ converge to 1 pointwise on $\kappa(\zeta, \delta)$ as $j \to \infty$, Fatou's Lemma shows that

$$\mu(\kappa(\zeta, \delta)) \le \liminf_{j \to \infty} \mu(\kappa(\zeta_j, \delta)).$$

In other words, the function $\zeta \mapsto \mu(\kappa(\zeta, \delta))$ is lower-semicontinuous on S. From the definition of $\mathcal{M}[\mu]$, we conclude that $\mathcal{M}[\mu]$ is the supremum of lower-semicontinuous functions on S, and thus $\mathcal{M}[\mu]$ is lower-semicontinuous. In particular, $\mathcal{M}[\mu] : S \to [0, \infty]$ is Borel measurable.

In the next theorem we begin to see the connection between the Hardy-Littlewood maximal function and the Fatou Theorem.

6.31 Theorem: *If $\mu \in M(S)$ and $u = P[\mu]$, then*

$$\mathcal{R}[u](\zeta) \le \mathcal{M}[\mu](\zeta)$$

for all $\zeta \in S$.

PROOF: Observe that if f is a continuous, positive, and increasing function on $[-1,1]$, then given $\varepsilon > 0$, there exists a step function

$$\varphi = c_0\chi_{[-1,1]} + \sum_{j=1}^{m} c_j\chi_{(t_j,1]}$$

such that $f \le \varphi \le f + \varepsilon$ on $[-1,1]$; here $-1 < t_1 < \cdots < t_m < 1$ and $c_0, \dots, c_m \in [0,\infty)$.

We may assume μ is positive and that $\zeta = \mathbf{N}$. Fix $r \in [0,1)$. Then $P(r\mathbf{N},\eta) = f(\eta_n)$, where

$$f(t) = \frac{1-r^2}{(1-2rt+r^2)^{n/2}}$$

for $t \in [-1,1]$. Let $\varepsilon > 0$. Because f has the properties specified in the first paragraph, there exists a step function φ as above with

$$P(r\mathbf{N},\eta) \le \varphi(\eta_n) \le P(r\mathbf{N},\eta) + \varepsilon$$

for all $\eta \in S$. Now for any $t \in \mathbf{R}$, the function on S defined by $\eta \mapsto \chi_{(t,1]}(\eta_n)$ is the characteristic function of an open cap centered at $\mathbf{N}$. We conclude that there are caps $\kappa_0, \dots, \kappa_m$, centered at $\mathbf{N}$, and nonnegative numbers $c_0, \dots, c_m$, such that

6.32 $$P(r\mathbf{N},\eta) \le \sum_{j=0}^{m} c_j\chi_{\kappa_j}(\eta) \le P(r\mathbf{N},\eta) + \varepsilon$$

for all $\eta \in S$.

Integrating the first inequality in 6.32 over S with respect to μ, we get

$$\begin{aligned} u(r\mathbf{N}) &\le \sum_{j=0}^{m} c_j\mu(\kappa_j) \\ &= \sum_{j=0}^{m} c_j\sigma(\kappa_j)(\mu(\kappa_j)/\sigma(\kappa_j)) \\ &\le \mathcal{M}[\mu](\mathbf{N})\Big(\sum_{j=0}^{m} c_j\sigma(\kappa_j)\Big) \\ &\le \mathcal{M}[\mu](\mathbf{N})(1+\varepsilon). \end{aligned}$$

The last inequality follows by integrating the second inequality in 6.32 over S with respect to σ. Because ε is arbitrary, we conclude that $u(r\mathrm{N}) \le \mathcal{M}[\mu](\mathrm{N})$, and thus $\mathcal{R}[u](\mathrm{N}) \le \mathcal{M}[\mu](\mathrm{N})$, as desired. ■

Theorem 6.37 below estimates the σ-measure of the set where $\mathcal{M}[\mu]$ is large. The "covering lemma" that we prove next will be a crucial ingredient in its proof. We abuse notation slightly and adopt the convention that if $\kappa = \kappa(\zeta, \delta)$, then 3κ denotes the cap $\kappa(\zeta, 3\delta)$.

6.33 Covering Lemma: *Given caps $\kappa_j = \kappa(\zeta_j, \delta_j)$, $j = 1, \dots, m$, there exists a subset $J \subset \{1, \dots, m\}$ such that:*

(a) *The collection $\{\kappa_j : j \in J\}$ is pairwise disjoint;*

(b) $\displaystyle\bigcup_{j=1}^{m} \kappa_j \subset \bigcup_{j \in J} 3\kappa_j$.

PROOF: We describe an inductive procedure for selecting the desired subcollection. Start by choosing a cap κ_{j_1} having the largest radius among the caps $\kappa_1, \dots, \kappa_m$. If all caps intersect κ_{j_1}, we stop. Otherwise, remove the caps intersecting κ_{j_1}, and from those remaining, select one of largest radius and denote it by κ_{j_2}. If all the remaining caps intersect κ_{j_2}, we stop; otherwise we continue as above. This process gives us a finite subcollection $\{\kappa_j : j \in J\}$, where $J = \{j_1, j_2, \dots\}$.

The subcollection $\{\kappa_j : j \in J\}$ clearly satisfies (a).

Given $\kappa \in \{\kappa_1, \dots, \kappa_m\}$, let κ' denote the first cap in the sequence $\kappa_{j_1}, \kappa_{j_2}, \dots$ such that $\kappa \cap \kappa'$ is nonempty. The way in which the caps in $\{\kappa_j : j \in J\}$ were chosen shows that the radius of κ' is at least as large as that of κ. By the triangle inequality, $\kappa \subset 3\kappa'$, proving (b). ■

In proving the next theorem we will need the fact that there exist constants $a > 0$, $A < \infty$, depending only on the dimension n, such that

6.34 $$a\delta^{n-1} \le \sigma(\kappa(\zeta, \delta)) \le A\delta^{n-1}$$

for all $\zeta \in S$ and all $\delta \in (0, 2]$. Intuitively, $\kappa(\zeta, \delta)$ looks like an $(n-1)$-dimensional ball of radius δ for small $\delta > 0$, indicating that 6.34 is correct. One may verify 6.34 rigorously by using formula A.3 in Appendix A.

From 6.34 we see that

6.35 $$\sigma(3\kappa) \le 3^{n-1}(A/a)\sigma(\kappa)$$

for all caps $\kappa \subset S$.

To motivate our next result, note that if $f \in L^1(S)$ and $t > 0$, then

$$t\sigma(\{|f| > t\}) \le \int_{\{|f|>t\}} |f|\, d\sigma \le \|f\|_1,$$

giving

6.36 $$\sigma(\{|f| > t\}) \le \frac{\|f\|_1}{t}.$$

Here we have used the abbreviated notation $\{|f| > t\}$ to denote the set $\{\zeta \in S : |f(\zeta)| > t\}$. The next theorem states that for $\mu \in M(S)$, the Hardy-Littlewood maximal function $\mathcal{M}[\mu]$ is almost in $L^1(S)$, in the sense that it satisfies an inequality resembling 6.36.

6.37 Theorem: *For every $\mu \in M(S)$ and every $t \in (0, \infty)$,*

$$\sigma(\{\mathcal{M}[\mu] > t\}) \le \frac{C\|\mu\|}{t},$$

where $C = 3^{n-1}(A/a)$.

PROOF: Suppose $t \in (0, \infty)$. Let $E \subset \{\mathcal{M}[\mu] > t\}$ be compact. Then for each $\zeta \in E$, there is a cap κ centered at ζ with $|\mu|(\kappa)/\sigma(\kappa) > t$. Being compact, E is covered by finitely many such caps. From these we may choose a subcollection with the properties specified in 6.33. Thus there are pairwise disjoint caps $\kappa_1, \dots, \kappa_N$ such that $3\kappa_1, \dots, 3\kappa_N$ cover E, and such that $|\mu|(\kappa_j)/\sigma(\kappa_j) > t$ for $j = 1, \dots, N$. By 6.35 and the definition of C, we therefore have

$$\sigma(E) \le \sum_{j=1}^{N} \sigma(3\kappa_j) \le C \sum_{j=1}^{N} \sigma(\kappa_j) \le C \sum_{j=1}^{N} \frac{|\mu|(\kappa_j)}{t} \le \frac{C\|\mu\|}{t};$$

the pairwise disjointness of the caps $\kappa_1, \dots, \kappa_N$ was used in the last inequality. Taking the supremum over all compact $E \subset \{\mathcal{M}[\mu] > t\}$ now gives the conclusion of the theorem. ∎

Let us write $\mathcal{M}[f]$ in place of $\mathcal{M}[\mu]$ when $d\mu = f\, d\sigma$ for $f \in L^1(S)$. The conclusion of Theorem 6.37 for $f \in L^1(S)$ is then

6.38 $$\sigma(\{\mathcal{M}[f] > t\}) \le \frac{C\|f\|_1}{t}.$$

We now prove the Fatou theorem for Poisson integrals of functions in $L^1(S)$.

6.39 Theorem: *If $f \in L^1(S)$, then $P[f]$ has nontangential limit $f(\zeta)$ at almost every $\zeta \in S$.*

PROOF: For $f \in L^1(S)$ and $\alpha > 0$, define the function $\mathcal{L}_\alpha[f]$ on S by

$$\mathcal{L}_\alpha[f](\zeta) = \limsup_{\substack{x \to \zeta \\ x \in \Omega_\alpha(\zeta)}} |P[f](x) - f(\zeta)|.$$

We first show that $\mathcal{L}_\alpha[f] = 0$ almost everywhere on S.

Note that

6.40 $$\mathcal{L}_\alpha[f] \le \mathcal{N}_\alpha[P[|f|]] + |f|,$$

and that $\mathcal{L}_\alpha[f_1+f_2] \le \mathcal{L}_\alpha[f_1]+\mathcal{L}_\alpha[f_2]$ (both statements holding almost everywhere on S). Note also that $\mathcal{L}_\alpha[f] \equiv 0$ for every $f \in C(S)$.

Now fix $f \in L^1(S)$ and $\alpha > 0$. Also fixing $t \in (0, \infty)$, our main goal is to show that $\sigma(\{\mathcal{L}_\alpha[f] > 2t\}) = 0$.

Given $\varepsilon > 0$, we may choose $g \in C(S)$ such that $\|f - g\|_1 < \varepsilon$. We then have

$$\begin{aligned}
\mathcal{L}_\alpha[f] &\le \mathcal{L}_\alpha[f - g] + \mathcal{L}_\alpha[g] \\
&= \mathcal{L}_\alpha[f - g] \\
&\le \mathcal{N}_\alpha[P[|f - g|]] + |f - g| \\
&\le C_\alpha \mathcal{R}[P[|f - g|]] + |f - g| \\
&\le C_\alpha \mathcal{M}[|f - g|] + |f - g|,
\end{aligned}$$

this holding at almost every point of S. In this string of inequalities we have used 6.40, 6.28, and 6.31 in succession.

We thus have

6.41 $$\{\mathcal{L}_\alpha[f] > 2t\} \subset \{C_\alpha \mathcal{M}[|f - g|] > t\} \cup \{|f - g| > t\}.$$

By 6.38 and 6.36, the σ-measure of the right side of 6.41 is less than or equal to

$$\frac{CC_\alpha \|f - g\|_1}{t} + \frac{\|f - g\|_1}{t}.$$

Recalling that $\|f - g\|_1 < \varepsilon$ and that ε is arbitrary, we have shown that the set $\{\mathcal{L}_\alpha[f] > 2t\}$ is contained in sets of arbitrarily small σ-measure, and therefore $\sigma(\{\mathcal{L}_\alpha[f] > 2t\}) = 0$.

Because this is true for every $t \in (0, \infty)$, we have proved $\mathcal{L}_\alpha[f] = 0$ almost everywhere on S.

To finish, let $f \in L^1(S)$, and define $E_m = \{\mathcal{L}_m[f] = 0\}$ for $m = 1, 2, \ldots$. We have shown that E_m is a set of full measure on S for each m, and thus $\bigcap E_m$ is a set of full measure. At each $\zeta \in \bigcap E_m$, $P[f]$ has nontangential limit $f(\zeta)$, which is what we set out to prove. ∎

Recall that $\mu \in M(S)$ is said to be *singular* with respect to σ, written $\mu \perp \sigma$, if there exists a Borel set $E \subset S$ such that $\sigma(E) = 0$ and $|\mu|(E) = \|\mu\|$. Recall also that each $\mu \in M(S)$ has a unique decomposition $d\mu = f\,d\sigma + d\mu_s$, where $f \in L^1(S)$ and $\mu_s \perp \sigma$; this is called the *Lebesgue decomposition* of μ with respect to σ. The following result is the second half of the Fatou Theorem for $h^1(B)$.

6.42 Theorem: *If $\mu \perp \sigma$, then $P[\mu]$ has nontangential limit 0 almost everywhere on S.*

PROOF: Much of the proof is similar to that of Theorem 6.39, and so we will be brief about certain details.

It suffices to prove the theorem for positive measures, so suppose $\mu \in M(S)$ is positive and $\mu \perp \sigma$. For $\alpha > 0$, define

$$\mathcal{L}_\alpha[\mu](\zeta) = \limsup_{\substack{x \to \zeta \\ x \in \Omega_\alpha(\zeta)}} P[\mu](x)$$

for $\zeta \in S$. Fixing $t \in (0, \infty)$, the proof will be completed by showing that $\sigma(\{\mathcal{L}_\alpha[\mu] > 2t\}) = 0$.

Let $\varepsilon > 0$. Because $\mu \perp \sigma$, the regularity of μ implies the existence of a compact set $K \subset S$ such that $\sigma(K) = 0$ and $\mu(S \setminus K) < \varepsilon$. Writing $\mu = \mu_1 + \mu_2$, with $d\mu_1 = \chi_K\,d\mu$ and $d\mu_2 = \chi_{S \setminus K}\,d\mu$, observe that $\mathcal{L}_\alpha[\mu_1] = 0$ on $S \setminus K$ (see Exercise 2 of this chapter) and that $\|\mu_2\| = \mu(S \setminus K) < \varepsilon$.

Because $\mathcal{L}_\alpha[\mu] \le \mathcal{L}_\alpha[\mu_1] + \mathcal{L}_\alpha[\mu_2]$, we have

6.43
$$\{\mathcal{L}_\alpha[\mu] > 2t\} \subset \{\mathcal{L}_\alpha[\mu_1] > t\} \cup \{\mathcal{L}_\alpha[\mu_2] > t\}$$
$$\subset K \cup \{C_\alpha \mathcal{M}[\mu_2] > t\}.$$

(The inequality $\mathcal{L}_\alpha[\mu_2] \le C_\alpha \mathcal{M}[\mu_2]$ is obtained as in the proof of Theorem 6.39.) Recalling that $\sigma(K) = 0$, we see by Theorem 6.37 that the left side of 6.43 is contained in a set of σ-measure at most $(CC_\alpha \|\mu_2\|)/t$, which is less than $(CC_\alpha \varepsilon)/t$. Since ε is arbitrary, we conclude that $\sigma(\{\mathcal{L}_\alpha[\mu] > 2t\}) = 0$, as desired. ∎

Theorems 6.39 and 6.42 immediately give the following result.

6.44 Corollary: *Suppose $\mu \in M(S)$ and $d\mu = f\,d\sigma + d\mu_s$ is the Lebesgue decomposition of μ with respect to σ. Then $P[\mu]$ has nontangential limit $f(\zeta)$ at almost every $\zeta \in S$.*

If $u \in h^1(B)$, then $u = P[\mu]$ for some $\mu \in M(S)$ by Theorem 6.13. Corollary 6.44 thus implies that u has nontangential limits almost everywhere on S. Because $h^p(B) \subset h^1(B)$ for all $p \in [1, \infty]$, the same result holds for all $u \in h^p(B)$.

Exercises

1. Show that if $f \in L^1(S)$ and $\zeta \in S$ is a point of continuity of f, then $P[f]$ extends continuously to $B \cup \{\zeta\}$.

2. Suppose $V \subset S$ is open, $\mu \in M(S)$, and $|\mu|(V) = 0$. Show that if $\zeta \in V$, then $P[\mu](x) \to 0$ as $x \to \zeta$ unrestrictedly in B.

3. Suppose that $\mu \in M(S)$ and $\zeta \in S$. Show that

$$(1-r)^{n-1}P[\mu](r\zeta) \to 2\mu(\{\zeta\})$$

as $r \to 1$.

4. Suppose that u is harmonic function on B and $0 \le r < s < 1$.

   (a) Prove that $\|u_r\|_1 = \|u_s\|_1$ if and only if there is a constant c such that $cu|_{sB}$ is positive.

   (b) Suppose $1 < p \le \infty$. Prove that $\|u_r\|_p = \|u_s\|_p$ if and only u is constant.

5. (a) Give an example of a normed linear space and a weak* convergent sequence in its dual space that is not norm-bounded.

   (b) Prove that in the dual space of a Banach space, every weak* convergent sequence is norm-bounded. (*Hint:* Use the uniform boundedness principle.)

6. It is easy to see that if $\mu_j \to \mu$ in $M(S)$, then $P[\mu_j] \to P[\mu]$ uniformly on compact subsets of B. Prove that the conclusion is still valid if we assume only that $\mu_j \to \mu$ weak* in $M(S)$.

7. Suppose that (μ_j) is a norm-bounded sequence in $M(S)$ such that $(P[\mu_j])$ converges pointwise on B. Prove that (μ_j) is weak* convergent in $M(S)$.

8. Prove directly (that is, without the help of Theorem 6.13) that $h^p(B)$ is a Banach space for every $p \in [1, \infty]$.

9. Prove that a real-valued function on B belongs to $h^1(B)$ if and only if it is the difference of two positive harmonic functions on B.

10. Let $\zeta \in S$. Show that $P(\cdot, \zeta) \in h^p(B)$ for $p = 1$ but not for any $p > 1$.

11. Suppose $1 < p < \infty$ and $u \in h^p(B)$. Prove that

$$(1 - |x|)^{(n-1)/p} u(x) \to 0$$

as $|x| \to 1$.

12. A family of functions $\mathcal{F} \subset L^1(S)$ is said to be *uniformly integrable* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\int_E |f| \, d\sigma < \varepsilon$ whenever $f \in \mathcal{F}$ and $\sigma(E) < \delta$. Show that a harmonic function u on B is the Poisson integral of a function in $L^1(S)$ if and only if the family $\{u_r : r \in [0, 1)\}$ is uniformly integrable.

13. Prove that there exists $u \in h^1(B)$ such that $u(B \cap B(\zeta, \varepsilon)) = \mathbf{R}$ for all $\zeta \in S$, $\varepsilon > 0$. (*Hint:* Let $u = P[\mu]$, where μ is a judiciously chosen sum of point masses.)

14. Suppose that $p \in [1, \infty)$ and u is harmonic on B. Show that $u \in h^p(B)$ if and only if there exists a harmonic function v on B such that $|u|^p \le v$ on B.

15. Suppose $n > 2$. Show that if u is positive and harmonic on $\{x \in \mathbf{R}^n : |x| > 1\}$, then there exists a unique positive measure $\mu \in M(S)$ and a unique nonnegative constant c such that

$$u(x) = P_e[\mu](x) + c(1 - |x|^{2-n})$$

for $|x| > 1$. (Here P_e is the external Poisson kernel defined in Chapter 4.) State and prove an analogous result for the case $n = 2$.

16. Let Ω denote B_3 minus the x_3-axis. Show that every bounded harmonic function on Ω extends to be harmonic on B_3.

17. Suppose $\zeta \in S$ and f is positive and continuous on $S \setminus \{\zeta\}$. Need there exist a positive harmonic function u on B that extends continuously to $\overline{B} \setminus \{\zeta\}$ with $u = f$ on $S \setminus \{\zeta\}$?

18. Suppose $E \subset S$ and $\sigma(E) = 0$. Prove that there exists a positive harmonic function u on B such that u has nontangential limit ∞ at every point of E.

19. Let $\mathcal{F}$ denote the family of all positive harmonic functions u on B such that $u(0) = 1$. Compute $\inf\{u(\mathbf{N}/2) : u \in \mathcal{F}\}$ and $\sup\{u(\mathbf{N}/2) : u \in \mathcal{F}\}$. Do there exist functions in $\mathcal{F}$ that attain either of these extreme values at $\mathbf{N}/2$? If so, are they unique?

20. Find all extreme points of $\mathcal{F}$, where $\mathcal{F}$ is the family defined in the previous exercise. (A function in $\mathcal{F}$ is called an *extreme point* of $\mathcal{F}$ if it cannot be written as the average of two distinct functions in $\mathcal{F}$.)

21. Show that

$$\int_S P(x,\zeta)P(y,\zeta)\,d\sigma(\zeta) = \frac{1 - |x|^2|y|^2}{(1 - 2x \cdot y + |x|^2|y|^2)^{n/2}}$$

for all $x, y \in B$.

22. Prove that if $u \in h^2(B)$, $u(0) = 0$, and $\|u\|_{h^2} \le 1$, then

$$|u(x)| \le \sqrt{\frac{1 + |x|^2}{(1 - |x|^2)^{n-1}} - 1}$$

for all $x \in B$.

23. Show that the bound on $|(\nabla u)(0)|$ given by Theorem 6.26 can fail if the requirement that u be real valued is dropped.

24. Show that

$$U_3(|x|\mathbf{N}) = \frac{1}{|x|}\left[1 - \frac{1 - |x|^2}{\sqrt{1 + |x|^2}}\right]$$

and

$$U_4(|x|\mathbf{N}) = \frac{2}{\pi}\,\frac{(1 + |x|^2)^2 \arctan|x| - |x|(1 - |x|^2)}{|x|^2(1 + |x|^2)}.$$

(*Hint:* Evaluate the Poisson integrals that define $U_3(|x|\mathbf{N})$ and $U_4(|x|\mathbf{N})$, using an appropriate result from Appendix A. Be prepared for some hard calculus.)

25. Suppose u is harmonic on B and $\sum_{m=0}^{\infty} p_m$ is the homogeneous expansion of u about 0. Prove that

$$\|u\|_{h^2} = \Big(\sum_{m=0}^{\infty} \int_S |p_m|^2\,d\sigma\Big)^{1/2}$$

26. *Schwarz Lemma for h^2-functions:* Prove that if u is harmonic on B and $\|u\|_{h^2} \le 1$, then $|\nabla u(0)| \le \sqrt{n}$. Find all functions for which this inequality is an equality.

27. For a smooth function u on B, we define the *radial derivative* $D_R u$ by setting $D_R u(x) = x \cdot \nabla u(x)$ for $x \in B$. Show that there exist positive constants c and C, depending only on the dimension n, such that

$$c\|u\|_{h^2} \le |u(0)| + \left(\int_B |D_R u(x)|^2 (1 - |x|)\, dV(x)\right)^{1/2} \le C\|u\|_{h^2}$$

for all u harmonic on B. (*Hint:* Use the homogeneous expansion of u, Exercise 29 in Chapter 1, and polar coordinates.)

28. (a) Find a measure $\mu \in M(S)$ with $\mathcal{M}[\mu] \notin L^1(S)$.

(b) Can the measure μ in part (a) be chosen to be absolutely continuous with respect to σ?

29. Let $\mu \in M(S)$. Show that if

$$\lim_{\delta \to 0} \frac{\mu(\kappa(\zeta, \delta))}{\sigma(\kappa(\zeta, \delta))} = L \in \mathbf{C},$$

then $\lim_{r \to 1} P[\mu](r\zeta) = L$. (*Suggestion:* Without loss of generality, $\zeta = \mathbf{N}$. For η near $\mathbf{N}$, approximate $P(r\mathbf{N}, \eta)$ as in the proof of Theorem 6.31.)

30. Let $f \in L^1(S)$. A point $\zeta \in S$ is called a *Lebesgue point* of f if

$$\lim_{\delta \to 0} \frac{1}{\sigma(\kappa(\zeta, \delta))} \int_{\kappa(\zeta,\delta)} |f - f(\zeta)|\, d\sigma = 0.$$

Show that almost every $\zeta \in S$ is a Lebesgue point of f. (*Hint:* Imitate the proof of Theorem 6.39.)

31. For u a function on B, let $u^*(\zeta)$ denote the nontangential limit of u at $\zeta \in S$, provided this limit exists. Show that if $1 < p \le \infty$ and $u \in h^p(B)$, then $u = P[u^*]$, while this need not hold if $u \in h^1(B)$.

32. Let $f(z) = e^{(1+z)/(1-z)}$ for $z \in B_2$. Show that the holomorphic function f has a nontangential limit with absolute value 1 at almost every point of S, even though f is unbounded on B_2. Explain why this does not contradict h^p-theory.

CHAPTER 7

Harmonic Functions on Half-Spaces

In this chapter we study harmonic functions defined on the upper half-space H. Harmonic function theory on H has a distinctly different flavor from that on B. One advantage of H over B is the dilation-invariance of H. We have already put this to good use in the section *Limits Along Rays* in Chapter 2. Some disadvantages that we will need to work around: ∂H is not compact, and Lebesgue measure on ∂H is not finite.

Recall that we identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$, writing a typical point $z \in \mathbf{R}^n$ as $z = (x, y)$, where $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$. The upper half-space $H = H_n$ is the set

$$H = \{(x, y) \in \mathbf{R}^n : y > 0\}.$$

We identify $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$; with this convention we then have $\partial H = \mathbf{R}^{n-1}$.

For u a function on H and $y > 0$, we let u_y denote the function on $\mathbf{R}^{n-1}$ defined by

$$u_y(x) = u(x, y).$$

The functions u_y play the same role on the upper half-space that the dilations play on the ball.

The Poisson Kernel for the Upper Half-Space

We seek a function P_H on $H \times \mathbf{R}^{n-1}$ analogous to the Poisson kernel for the ball. Thus, for each fixed $t \in \mathbf{R}^{n-1}$, we would like $P_H(\cdot, t)$ to be a positive harmonic function on H having the appropriate approximate-identity properties (see 1.20).

Fix $t = 0$ temporarily; we will concentrate first on finding $P_H(\cdot, 0)$. Taking our cue from Theorem 6.19, we look for a positive harmonic function on H that extends continuously to $\overline{H} \setminus \{0\}$ with boundary values 0 on $\mathbf{R}^{n-1} \setminus \{0\}$. One such function is $u(x, y) = y$, but obviously this is not what we want—u doesn't "blow up" at 0 as we know $P_H(\cdot, 0)$ should. On the other hand, u does blow up at ∞. Applying the Kelvin transform, we can move the singularity of u from ∞ to 0 and arrive at the desired function.

Thus, with $u(x, y) = y$, let us define

$$v = K[u]$$

on H, where K is the Kelvin transform introduced in Chapter 4. A simple computation shows that

$$v(x, y) = \frac{y}{(|x|^2 + y^2)^{n/2}}$$

for all $(x, y) \in H$. Because the inversion map preserves the upper half-space and the Kelvin transform preserves harmonic functions, we know without any computation that v is a positive harmonic function on H.

The function v has the property that $v_y(x) = y^{-(n-1)} v_1(x/y)$ for all $(x, y) \in H$. Therefore the change of variables $x \mapsto yx'$ shows that $\int_{\mathbf{R}^{n-1}} v_y(x)dx$ is the same for all $y > 0$. (Here dx denotes Lebesgue measure on $\mathbf{R}^{n-1}$.) Because $\int_{\mathbf{R}^{n-1}} v_1(x)\,dx < \infty$ (verify using polar coordinates—see 1.5), there exists a positive constant c_n such that

$$c_n \int_{\mathbf{R}^{n-1}} v_y(x)\,dx = c_n \int_{\mathbf{R}^{n-1}} \frac{y}{(|x|^2 + y^2)^{n/2}}\,dx = 1$$

for all $y > 0$. We will show that $c_n = 2/(nV(B_n))$ at the end of this section.

The function $c_n v$ has all the properties we sought for $P_H(\cdot, 0)$. To obtain $P_H(\cdot, t)$, we simply translate $c_n v$ by t. Thus we now make our official definition: for $z = (x, y) \in H$ and $t \in \mathbf{R}^{n-1}$, set

$$P_H(z,t) = c_n \frac{y}{(|x-t|^2 + y^2)^{n/2}}.$$

The function P_H is called the *Poisson kernel* for the upper half-space.

Note that P_H can be written as

$$P_H(z,t) = c_n \frac{y}{|z-t|^n}.$$

In this form, P_H reminds us of the Poisson kernel for the ball. (If $(x, y) \in H$, then y is the distance from (x, y) to ∂H; analogously, the numerator of the Poisson kernel for B is roughly the distance to ∂B.)

The work above shows that $P_H(\cdot, t)$ is positive and harmonic on H for each $t \in \mathbf{R}^{n-1}$. We have also seen that

7.1
$$\int_{\mathbf{R}^{n-1}} P_H(z,t)\, dt = 1$$

for each $z \in H$. The next result gives the remaining approximate-identity property that we need to solve the Dirichlet problem for H.

7.2 Proposition: *For every $a \in \mathbf{R}^{n-1}$ and every $\delta > 0$,*

$$\int_{|t-a|>\delta} P_H(z,t)\, dt \to 0$$

as $z \to a$.

We leave the proof of Proposition 7.2 to the reader; it follows without difficulty from the definition of P_H.

Let us now evaluate the normalizing constant c_n. We accomplish this with a slightly underhanded trick:

$$\begin{aligned}
\frac{1}{c_n} &= \frac{2}{\pi}\int_0^\infty \frac{1}{1+y^2}\int_{\mathbf{R}^{n-1}} \frac{y}{(|x|^2+y^2)^{n/2}}\, dx\, dy \\
&= \frac{2}{\pi}\int_H \frac{y}{(1+y^2)(|x|^2+y^2)^{n/2}}\, dx\, dy \\
&= \frac{2nV(B)}{\pi}\int_{S^+}\int_0^\infty \frac{\zeta_n}{1+(r\zeta_n)^2}\, dr\, d\sigma(\zeta) \\
&= nV(B)/2,
\end{aligned}$$

where the third equality is obtained by switching to polar coordinates (see 1.5) and S^+ denotes the upper half-sphere.

The Dirichlet Problem for the Upper Half-Space

For μ a complex Borel measure on $\mathbf{R}^{n-1}$, the Poisson integral of μ, denoted by $P_H[\mu]$, is the function on H defined by

$$P_H[\mu](z) = \int_{\mathbf{R}^{n-1}} P_H(z,t)\,d\mu(t).$$

We can verify that $P_H[\mu]$ is harmonic on H by differentiating under the integral sign, or by noting that $P_H[\mu]$ satisfies the volume version of the mean-value property on H.

We let $M(\mathbf{R}^{n-1})$ denote the set of complex Borel measures on $\mathbf{R}^{n-1}$. With the total variation norm $\|\ \|$, the Banach space $M(\mathbf{R}^{n-1})$ is the dual space of $C_0(\mathbf{R}^{n-1})$, the space of continuous functions f on $\mathbf{R}^{n-1}$ that vanish at ∞ (equipped with the supremum norm).

For $1 \le p < \infty$, $L^p(\mathbf{R}^{n-1})$ denotes the space of Borel measurable functions f on $\mathbf{R}^{n-1}$ for which

$$\|f\|_p = \left(\int_{\mathbf{R}^{n-1}} |f(x)|^p\,dx\right)^{1/p} < \infty;$$

$L^\infty(\mathbf{R}^{n-1})$ consists of the Borel measurable functions f on $\mathbf{R}^{n-1}$ for which $\|f\|_\infty < \infty$, where $\|f\|_\infty$ denotes the essential supremum norm on $\mathbf{R}^{n-1}$ with respect to Lebesgue measure.

Recall that on S, if $p > q$ then $L^p(S) \subset L^q(S)$. On $\mathbf{R}^{n-1}$, if $p \ne q$ then neither of the spaces $L^p(\mathbf{R}^{n-1})$, $L^q(\mathbf{R}^{n-1})$ contains the other. The reader should keep this in mind as we develop Poisson-integral theory in this new setting.

The Poisson integral of $f \in L^p(\mathbf{R}^{n-1})$, for any $p \in [1,\infty]$, is the function $P_H[f]$ on H defined by

$$P_H[f](z) = \int_{\mathbf{R}^{n-1}} f(t)P_H(z,t)\,dt.$$

Because $P_H(z,\cdot)$ belongs to $L^q(\mathbf{R}^{n-1})$ for every $q \in [1,\infty]$, the integral above is well-defined for every $z \in H$ (by Hölder's inequality). An

argument like the one given for $P_H[\mu]$ shows that $P_H[f]$ is harmonic on H.

We now prove a result that the reader has surely already guessed.

7.3 Solution of the Dirichlet Problem for H: *Suppose f is continuous and bounded on $\mathbf{R}^{n-1}$. Define u on $\overline{H}$ by*

$$u(z) = \begin{cases} P_H[f](z) & \text{if } z \in H \\ f(z) & \text{if } z \in \mathbf{R}^{n-1}. \end{cases}$$

Then u is continuous on $\overline{H}$ and harmonic on H. Moreover,

$$|u| \le \|f\|_\infty$$

on $\overline{H}$.

PROOF: The estimate $|u| \le \|f\|_\infty$ on $\overline{H}$ is immediate from 7.1. We already know that u is harmonic on H.

The proof that u is continuous on $\overline{H}$ is like that of Theorem 1.17. Specifically, let $a \in \mathbf{R}^{n-1}$ and $\delta > 0$. Then

$$\begin{aligned} |u(z) - f(a)| &= \left| \int_{\mathbf{R}^{n-1}} (f(t) - f(a)) P_H(z,t)\, dt \right| \\ &\le \int_{|t-a|\le\delta} |f(t) - f(a)|\, P_H(z,t)\, dt \\ &\qquad + 2\|f\|_\infty \int_{|t-a|>\delta} P_H(z,t)\, dt \end{aligned}$$

for all $z \in H$. If δ is small, the integral over $\{|t-a| \le \delta\}$ will be small by the continuity of f at a and 7.1. The integral over $\{|t-a| > \delta\}$ approaches 0 as $z \to a$ by Proposition 7.2. ∎

In the special case where f is uniformly continuous on $\mathbf{R}^{n-1}$, we can make a stronger assertion:

7.4 Theorem: *If f is bounded and uniformly continuous on $\mathbf{R}^{n-1}$ and $u = P_H[f]$, then $u_y \to f$ uniformly on $\mathbf{R}^{n-1}$ as $y \to 0$.*

PROOF: The uniform continuity of f on $\mathbf{R}^{n-1}$ shows that the estimates in the proof of Theorem 7.3 can be made uniformly in a. ∎

See Exercise 4 of this chapter for a converse to Theorem 7.4.

The next result follows immediately from Corollary 2.2 and Theorem 7.3; we state it as a theorem because of its importance.

7.5 Theorem: *Suppose u is a continuous bounded function on $\overline{H}$ that is harmonic on H. Then u is the Poisson integral of its boundary values. More precisely,*

$$u = P_H[u|_{\mathbf{R}^{n-1}}]$$

on H.

We now take up the more general Poisson integrals defined earlier. Certain statements and proofs closely parallel those in the last chapter; we will be brief about details in these cases.

7.6 Theorem: *The following growth estimates apply to Poisson integrals:*

(a) *If $\mu \in M(\mathbf{R}^{n-1})$ and $u = P_H[\mu]$, then $\|u_y\|_1 \le \|\mu\|$ for all $y > 0$.*

(b) *Suppose $1 \le p \le \infty$. If $f \in L^p(\mathbf{R}^{n-1})$ and $u = P_H[f]$, then $\|u_y\|_p \le \|f\|_p$ for all $y > 0$.*

PROOF: The identity

7.7 $$P_H((x,y),t) = P_H((t,y),x),$$

valid for all $x, t \in \mathbf{R}^{n-1}$ and $y > 0$, is the replacement for 6.5 in this context. The rest of the proof is the same as that of Theorem 6.4. ∎

The next result is the upper half-space analogue of Theorem 6.7. Here the noncompactness of $\partial H = \mathbf{R}^{n-1}$ forces us to do a little extra work.

7.8 Theorem: *Suppose that $1 \le p < \infty$. If $f \in L^p(\mathbf{R}^{n-1})$ and $u = P_H[f]$, then $\|u_y - f\|_p \to 0$ as $y \to 0$.*

PROOF: We first prove the theorem for $f \in C_c(\mathbf{R}^{n-1})$, the set of continuous functions on $\mathbf{R}^{n-1}$ with compact support. Because $C_c(\mathbf{R}^{n-1})$ is dense in $L^p(\mathbf{R}^{n-1})$ for $1 \le p < \infty$, the approximation argument used in proving Theorem 6.7 (together with Theorem 7.6) will finish the proof.

Let $f \in C_c(\mathbf{R}^{n-1})$, and set $u = P_H[f]$. Choose a ball $B(0, R)$ that contains the support of f. Because f is uniformly continuous on $\mathbf{R}^{n-1}$, Theorem 7.4 implies that $u_y \to f$ uniformly on $\mathbf{R}^{n-1}$ as $y \to 0$. Thus to show that $\|u_y - f\|_p \to 0$, we need only show that

7.9
$$\int_{|x|>2R} |u_y(x)|^p \, dx \to 0$$

as $y \to 0$.

For $|x| > 2R$, we have

$$\begin{aligned} |u_y(x)|^p &\le \int_{|t|<R} |f(t)|^p \frac{c_n y}{(|x-t|^2 + y^2)^{n/2}} \, dt \\ &\le \frac{Cy}{(|x| - R)^n}, \end{aligned}$$

where $C = c_n \|f\|_\infty^p V_{n-1}(B(0,R))$ and the first inequality follows from Jensen's inequality. It is now easy to verify 7.9 by integrating in polar coordinates (1.5). ∎

As in the last chapter, weak* convergence replaces norm convergence for Poisson integrals of measures and L^∞-functions.

7.10 Theorem: *Poisson integrals have the following weak* convergence properties:*

(a) *If $\mu \in M(\mathbf{R}^{n-1})$ and $u = P_H[\mu]$, then $u_y \to \mu$ weak* in $M(\mathbf{R}^{n-1})$ as $y \to 0$.*

(b) *If $f \in L^\infty(\mathbf{R}^{n-1})$ and $u = P_H[f]$, then $u_y \to f$ weak* in $L^\infty(\mathbf{R}^{n-1})$ as $y \to 0$.*

PROOF: The Banach spaces $M(\mathbf{R}^{n-1})$ and $L^\infty(\mathbf{R}^{n-1})$ are, respectively, the dual spaces of $C_0(\mathbf{R}^{n-1})$ and $L^1(\mathbf{R}^{n-1})$. Note that if $g \in C_0(\mathbf{R}^{n-1})$, then g is uniformly continuous on $\mathbf{R}^{n-1}$, and therefore $(P_H[g])_y \to g$ uniformly on $\mathbf{R}^{n-1}$ as $y \to 0$ (by Theorem 7.4). The proof of Theorem 6.9 can thus be used here, essentially verbatim. Again, the identity 7.7 replaces 6.5. ∎

If f is continuous and bounded on $\mathbf{R}^{n-1}$, then there is a function on the closed half-space $\overline{H}$ that is harmonic on H and agrees with f on the boundary $\mathbf{R}^{n-1}$; see 7.3. What happens if we drop the assumption that

f is bounded? Without any growth estimate on f, we cannot expect to find a solution to the Dirichlet problem with boundary data f by integrating f against some kernel as in 7.3. Nevertheless, the following theorem asserts that a solution exists with just the assumption that f is continuous. We are not asserting any sort of uniqueness for the solution, because a multiple of y can be added to any solution to obtain another solution.

7.11 Theorem: *Suppose $f \in C(\mathbf{R}^{n-1})$. Then there exists $u \in C(\overline{H})$ such that u is harmonic on H and $u|_{\mathbf{R}^{n-1}} = f$.*

PROOF: We will construct a sequence of functions $u_0, u_1, \dots$ in $C(\overline{H})$ such that for each k the following hold:

(a) u_k is harmonic on H;

(b) $u_k(x, 0) = f(x)$ for all $x \in \mathbf{R}^{n-1}$ with $|x| \le k$;

(c) $|(u_{k+1} - u_k)(x, y)| < \dfrac{1}{2^k}$ for all $(x, y) \in \overline{H}$ with $|(x, y)| \le k/2$.

This will prove the theorem, because (c) implies that the sequence (u_k) converges uniformly on each compact subset of $\overline{H}$ to a function $u \in C(\overline{H})$; from (a) and Theorem 1.23 we have that u is harmonic on H; from (b) we have that $u|_{\mathbf{R}^{n-1}} = f$.

We construct the sequence (u_k) inductively, starting by taking u_0 to be the constant function whose value is $f(0)$. Now fix k and suppose that we have $u_k \in C(\overline{H})$ satisfying (a) and (b) above. To construct u_{k+1}, let $w \in C(\overline{H})$ be such that w is harmonic on H and $w(x, 0) = f(x)$ for all $x \in \mathbf{R}^{n-1}$ with $|x| \le k + 1$ (to see that such a w exists, extend $f|_{(k+1)B_{n-1}}$ to a bounded continuous function on $\mathbf{R}^{n-1}$ and then use 7.3). Now

$$(w - u_k)(x, 0) = 0$$

for all $x \in \mathbf{R}^{n-1}$ with $|x| \le k$. Thus by the Schwarz reflection principle (4.12), $(w - u_k)|_{kB \cap \overline{H}}$ extends to a harmonic function v on kB. The proof of 4.12 shows that

7.12 $$v(x, y) = -v(x, -y)$$

for all $(x, y) \in kB$.

The expansion of v into an infinite sum of homogeneous harmonic polynomials converges uniformly to v on $(k/2)B$ (see 5.34). Taking an appropriate partial sum, we conclude that there is a harmonic polynomial p such that

$$|(v-p)(x,y)| < \frac{1}{2^k}$$

for all $(x,y) \in (k/2)B$. Note that $p(x,0) = 0$ for all $x \in \mathbf{R}^{n-1}$, because 7.12 implies that the power series expansion of v contains only odd powers of y.

Now let $u_{k+1} = w - p|_{\overline{H}}$. Then $u_{k+1} \in C(\overline{H})$ and u_{k+1} is harmonic on H. Furthermore, $u_{k+1}(x,0) = f(x)$ for all $x \in \mathbf{R}^{n-1}$ with $|x| \le k+1$. Finally, if $(x,y) \in \overline{H}$ with $|(x,y)| \le k/2$, then

$$\begin{aligned} |(u_{k+1} - u_k)(x,y)| &= |(w - p - u_k)(x,y)| \\ &= |(v-p)(x,y)| \\ &< \frac{1}{2^k} \end{aligned}$$

and thus u_{k+1} has all the desired properties. ∎

The Harmonic Hardy Spaces $h^p(H)$

For $p \in [1,\infty]$, we define the harmonic Hardy space $h^p(H)$ to be the normed vector space of functions u harmonic on H for which

$$\|u\|_{h^p} = \sup_{y>0} \|u_y\|_p < \infty.$$

Note that $h^\infty(H)$ is simply the collection of bounded harmonic functions on H, and that

$$\|u\|_{h^\infty} = \sup_{z\in H} |u(z)|.$$

We leave it to the reader to verify that $h^p(H)$ is a normed linear space under the norm $\|\ \|_{h^p}$.

As the reader should suspect by considering what happens in the ball, if $u \in h^p(H)$ then the norms $\|u_y\|_p$ increase as $y \to 0$ to $\|u\|_{h^p}$. To prove this, we need to do some extra work because of the noncompactness of $\partial H = \mathbf{R}^{n-1}$. We begin with the following result.

7.13 Theorem: *Let $p \in [1, \infty)$. Then there exists a constant C, depending only on p and n, such that*

$$|u(x,y)| \le \frac{C\|u\|_{h^p}}{y^{(n-1)/p}}$$

for all $u \in h^p(H)$ and all $(x,y) \in H$. In particular, every $u \in h^p(H)$ is bounded on $H + (0,y)$ for each $y > 0$.

PROOF: Let $(x_0, y_0) \in H$, and let ω denote the open ball in $\mathbf{R}^n$ with center (x_0, y_0) and radius $y_0/2$. The volume version of the mean-value property, together with Jensen's inequality, shows that

7.14
$$\begin{aligned}|u(x_0,y_0)|^p &\le \frac{1}{V_n(\omega)}\int_\omega |u|^p\,dV_n\\ &= \frac{2^n}{V_n(B){y_0}^n}\int_\omega |u|^p\,dV_n.\end{aligned}$$

Setting $\Omega = \{(x,y) \in H : y_0/2 < y < 3y_0/2\}$, we have

$$\begin{aligned}\int_\omega |u|^p\,dV_n &\le \int_\Omega |u|^p\,dV_n\\ &= \int_{y_0/2}^{3y_0/2}\int_{\mathbf{R}^{n-1}} |u(x,y)|^p\,dx\,dy\\ &\le y_0(\|u\|_{h^p})^p.\end{aligned}$$

This estimate and 7.14 give the conclusion of the theorem after taking p^{th} roots. ■

Theorems 7.5 and 7.13 show that if $p \in [1, \infty]$ and $u \in h^p(H)$, then for each $y > 0$ we have

7.15
$$u(z + (0,y)) = P_H[u_y](z)$$

for all $z \in H$.

The next corollary is not entirely analogous to Corollary 6.6 because the conclusion that $\|u_y\|_p$ increases as y decreases is not true for an arbitrary harmonic function on H. For example, if $u(x,y) = y - 1$, then $\|u_1\|_p = 0$ while $\|u_y\|_p = \infty$ for all $y \ne 1$. Thus we have the hypothesis in the next corollary that $u \in h^p(H)$.

7.16 Corollary: *Suppose* $1 \le p \le \infty$ *and* $u \in h^p(H)$. *Then*

$$\|u_{y_2}\|_p \le \|u_{y_1}\|_p$$

whenever $0 < y_1 \le y_2$. *Furthermore,*

$$\|u\|_{h^p} = \lim_{y \to 0} \|u_y\|_p.$$

PROOF: The idea is the same as in the proof of Corollary 6.6. Specifically, if $0 < y_1 \le y_2$ then

$$\|u_{y_2}\|_p = \|P[u_{y_1}]_{y_2 - y_1}\|_p \le \|u_{y_1}\|_p,$$

where the equality follows from 7.15 and the inequality follows from Theorem 7.6(b).

The formula for $\|u\|_{h^p}$ now immediately follows from the definition of $\|u\|_{h^p}$ and the first part of the corollary. ■

The next theorem is the analogue for the half-space of Theorem 6.13 for the ball. The results we have proved so far in this chapter allow the proof from the ball to carry over directly to the half-space, as the reader should verify.

7.17 Theorem: *The Poisson integral induces the following surjective isometries:*

(a) *The map* $\mu \mapsto P_H[\mu]$ *is a linear isometry of* $M(\mathbf{R}^{n-1})$ *onto* $h^1(H)$.

(b) *For* $1 < p \le \infty$, *the map* $f \mapsto P_H[f]$ *is a linear isometry of* $L^p(\mathbf{R}^{n-1})$ *onto* $h^p(H)$.

From the Ball to the Upper Half-Space, and Back

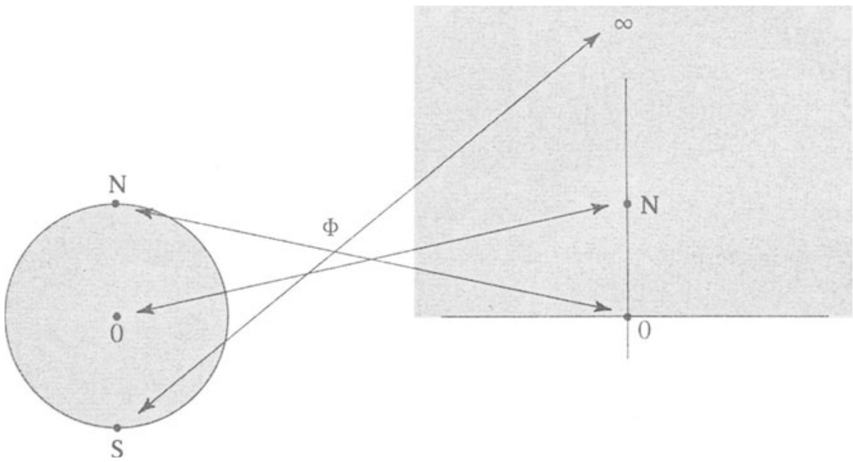
Recall the inversion map $x \mapsto x^*$ defined in Chapter 4. This map takes spheres containing 0 onto hyperplanes, and takes the interiors of such spheres onto open half-spaces. Composing the inversion map with appropriate translations and dilations will give us a one-to-one

map of B onto H. There are many such maps; the one we choose below has the advantage of being its own inverse under composition.

Let $\mathbf{N} = (0, 1)$ and $\mathbf{S} = (0, -1)$ (here 0 denotes the origin in $\mathbf{R}^{n-1}$); we can think of $\mathbf{N}$ and $\mathbf{S}$ as the north and south poles of the unit sphere S. Now define $\Phi: \mathbf{R}^n \setminus \{\mathbf{S}\} \to \mathbf{R}^n \setminus \{\mathbf{S}\}$ by

$$\Phi(z) = 2(z - \mathbf{S})^* + \mathbf{S}.$$

It is easy to see that Φ is a one-to-one map of $\mathbf{R}^n \setminus \{\mathbf{S}\}$ onto itself. We can regard Φ as a homeomorphism of $\mathbf{R}^n \cup \{\infty\}$ onto itself by defining $\Phi(\mathbf{S}) = \infty$ and $\Phi(\infty) = \mathbf{S}$. The reader may find it helpful to keep the following diagram in mind as we proceed.



Φ maps B onto H and H onto B.

The next result summarizes the basic properties of Φ.

7.18 Proposition: *The map Φ has the following properties:*

(a) *$\Phi(\Phi(z)) = z$ for all $z \in \mathbf{R}^n \cup \{\infty\}$;*

(b) *Φ is a conformal, one-to-one map of $\mathbf{R}^n \setminus \{\mathbf{S}\}$ onto $\mathbf{R}^n \setminus \{\mathbf{S}\}$;*

(c) *Φ maps B onto H and H onto B;*

(d) *Φ maps $S \setminus \{\mathbf{S}\}$ onto $\mathbf{R}^{n-1}$ and $\mathbf{R}^{n-1}$ onto $S \setminus \{\mathbf{S}\}$.*

PROOF: The proof of (a) is a simple computation.

In (b), only conformality needs to be checked. Recalling that the inversion map is conformal (Proposition 4.2), we see that Φ is the composition of conformal maps, and hence is itself conformal.

We prove (c) and (d) together. Noting that $\Phi(\mathbf{S}) = \infty$, we know that Φ maps $S \setminus \{\mathbf{S}\}$ onto some hyperplane. Because the inversion map preserves the $(0, y)$-axis, the same is true Φ. The conformality of Φ thus shows that $\Phi(S \setminus \{\mathbf{S}\})$ is a hyperplane perpendicular to the $(0, y)$-axis. Since $\Phi(\mathbf{N}) = 0$, we must have $\Phi(S \setminus \{\mathbf{S}\}) = \mathbf{R}^{n-1}$. It follows that $\Phi(B)$ is either the upper or lower half-space. Because $\Phi(0) = \mathbf{N}$, we have $\Phi(B) = H$, as desired. ∎

We now introduce a modified Kelvin transform $\mathcal{K}$ that will take harmonic functions on B to harmonic functions on H and vice-versa. Given any function u defined on a set $E \subset \mathbf{R}^n \setminus \{\mathbf{S}\}$, we define the function $\mathcal{K}[u]$ on $\Phi(E)$ by

$$\mathcal{K}[u](z) = 2^{(n-2)/2}|z - \mathbf{S}|^{2-n}u(\Phi(z)).$$

Note that when $n = 2$, $\mathcal{K}[u](z) = u(\Phi(z))$.

The factor $2^{(n-2)/2}$ is included so that $\mathcal{K}$ will be its own inverse. That is, we claim

$$\mathcal{K}[\mathcal{K}[u]] = u$$

for all u as above, a computation we leave to the reader.

The transform $\mathcal{K}$ is linear—if u, v are functions on E and b, c are constants, then

$$\mathcal{K}[bu + cv] = b\mathcal{K}[u] + c\mathcal{K}[v]$$

on $\Phi(E)$.

Finally, $\mathcal{K}$ preserves harmonicity. The real work for the proof of this was done when we proved Theorem 4.7.

7.19 Proposition: *If $\Omega \subset \mathbf{R}^n \setminus \{\mathbf{S}\}$, then u is harmonic on Ω if and only if $\mathcal{K}[u]$ is harmonic on $\Phi(\Omega)$.*

PROOF: Because $\mathcal{K}$ is its own inverse, it suffices to prove only one direction of the theorem. So suppose that u is harmonic on Ω. Define

a harmonic function v on $\frac{1}{2}(\Omega - \mathbf{S})$ by $v(z) = u(2z + \mathbf{S})$. By Theorem 4.7, the Kelvin transform $K[v]$ is harmonic on $2(\Omega - \mathbf{S})^*$, and thus $K[v](z - \mathbf{S})$ is harmonic on $2(\Omega - \mathbf{S})^* + \mathbf{S} = \Phi(\Omega)$. But, as is easily checked, $K[v](z - \mathbf{S}) = 2^{(2-n)/2}\mathcal{K}[u](z)$, so that $\mathcal{K}[u]$ is harmonic on $\Phi(\Omega)$, as desired. ∎

Positive Harmonic Functions on the Upper Half-Space

Because the modified Kelvin transform $\mathcal{K}$ takes positive functions to positive functions, Proposition 7.19 shows that $\mathcal{K}$ preserves the class of positive harmonic functions. Thus u is positive and harmonic on H if and only if $\mathcal{K}[u]$ is positive and harmonic on B. This will allow us to transfer our knowledge about positive harmonic functions on the ball to the upper half-space. For example, we can now prove an analogue of Theorem 6.19 for the upper half-space.

7.20 Theorem: *Let $t \in \mathbf{R}^{n-1}$. Suppose that u is positive and harmonic on H, and that u extends continuously to $\overline{H} \setminus \{t\}$ with boundary values 0 on $\mathbf{R}^{n-1} \setminus \{t\}$. Suppose further that*

7.21
$$\frac{u(0,y)}{y} \to 0 \quad \text{as } y \to \infty.$$

Then $u = cP_H(\cdot, t)$ for some positive constant c.

PROOF: The function $\mathcal{K}[u]$ is positive and harmonic on B. Thus by 6.19,

$$\mathcal{K}[u] = P[\mu]$$

for some positive $\mu \in M(S)$, where as usual P denotes the Poisson kernel for the ball. Our hypothesis on u implies that $\mathcal{K}[u]$ extends continuously to $\overline{B} \setminus \{\mathbf{S}, \Phi(t)\}$, with boundary values 0 on $S \setminus \{\mathbf{S}, \Phi(t)\}$. The argument used in proving Theorem 6.19 then shows that μ is the sum of point masses at $\mathbf{S}$ and $\Phi(t)$.

An easy computation gives

$$\mathcal{K}[u](r\mathbf{S}) = 2^{(n-2)/2}(1-r)^{2-n}u\Big(0, \frac{1+r}{1-r}\Big)$$

for every $r \in [0,1)$. Now from 7.21 we see that $(1-r)^{n-1}\mathcal{K}[u](r\mathbf{S}) \to 0$ as $r \to 1$, and this implies $\mu(\{\mathbf{S}\}) = 0$ (see Exercise 3 in Chapter 6).

Thus μ is a point mass at $\Phi(t)$, and therefore $\mathcal{K}[u]$ is a constant times $P(\cdot, \Phi(t))$. Because $P_H(\cdot, t)$ also satisfies the hypotheses of Theorem 7.20, $\mathcal{K}[P_H(\cdot, t)]$ is a constant times $P(\cdot, \Phi(t))$ as well. Thus

$$\mathcal{K}[u] = c\mathcal{K}[P_H(\cdot, t)]$$

for some positive constant c. Applying $\mathcal{K}$ to both sides of the last equation, we see that the linearity of $\mathcal{K}$ gives the conclusion of the theorem. ■

We can think of the next result as the "$t = \infty$" case of Theorem 7.20.

7.22 Theorem: *Suppose that u is positive and harmonic on H and that u extends continuously to $\overline{H}$ with boundary values 0 on $\mathbf{R}^{n-1}$. Then there exists a positive constant c such that $u(x, y) = cy$ for all $(x, y) \in H$.*

PROOF: The function $\mathcal{K}[u]$ is positive and harmonic on B, extends continuously to $\overline{B} \setminus \{\mathbf{S}\}$, and has boundary values 0 on $S \setminus \{\mathbf{S}\}$. By Theorem 6.19, $\mathcal{K}[u]$ is a constant times $P(\cdot, \mathbf{S})$. Because the same is true of $\mathcal{K}[v]$, where $v(x, y) = y$ on H, $\mathcal{K}[u]$ is a constant times $\mathcal{K}[v]$. As in the proof of the last theorem, this gives us the desired conclusion. ■

The modified Kelvin transform $\mathcal{K}$ allows us to derive the relation between P and P_H, the Poisson kernels for B and H, with a minimum of computation.

7.23 Theorem:

$$P_H(z, t) = 2^{n-2}c_n(1 + |t|^2)^{-n/2}|z - \mathbf{S}|^{2-n}P(\Phi(z), \Phi(t))$$

for all $z \in H$ and $t \in \mathbf{R}^{n-1}$.

PROOF: Fix $t \in \mathbf{R}^{n-1}$, and let $u(z)$ denote the right side of the equation above that we want to prove. Then u is positive and harmonic on H, and it is easy to check that u extends continuously to $\overline{H} \setminus \{t\}$ with boundary values 0 on $\mathbf{R}^{n-1} \setminus \{t\}$. We also see that $u(0, y)/y \to 0$ (with plenty of room to spare) as $y \to \infty$. Thus by Theorem 7.20, u is a constant multiple of $P_H(\cdot, t)$. Evaluating at $z = \mathbf{N}$ now gives the desired result. ■

We turn now to the problem of characterizing the positive harmonic functions on H. We know that if μ is a finite positive Borel measure on $\mathbf{R}^{n-1}$, then $P_H[\mu]$ is a positive harmonic function on H. Unlike the case for the ball, however, not all positive harmonic functions on H arise in this manner. In the first place, $P_H[\mu]$ defines a positive harmonic function on H for some positive measures μ that are not finite—Lebesgue measure on $\mathbf{R}^{n-1}$, for example. Secondly, the positive harmonic function y is not the Poisson integral of anything that lives on $\mathbf{R}^{n-1}$.

Let us note that if μ is any positive Borel measure on $\mathbf{R}^{n-1}$, then

7.24
$$P_H[\mu](z) = \int_{\mathbf{R}^{n-1}} P_H(z,t)\,d\mu(t)$$

is well-defined as a number in $[0,\infty]$ for every $z \in H$. We claim that 7.24 defines a positive harmonic function on H precisely when

7.25
$$\int_{\mathbf{R}^{n-1}} \frac{d\mu(t)}{(1+|t|^2)^{n/2}} < \infty.$$

To see this, note that if $z \in H$ is fixed, then $P_H(z,t)$, as a function of t, is bounded above and below by positive constant multiples of $(1+|t|^2)^{-n/2}$. Thus if 7.24 is finite for some $z \in H$, then it is finite for all $z \in H$, and this happens exactly when 7.25 occurs. In this case $P_H[\mu]$ is harmonic on H, as can be verified by checking the volume version of the mean-value property.

We now state the main result of this section.

7.26 Theorem: *If u is positive and harmonic on H, then there exists a positive Borel measure μ on $\mathbf{R}^{n-1}$ and a nonnegative constant c such that*
$$u(x,y) = P_H[\mu](x,y) + cy$$
for all $(x,y) \in H$.

The main idea in the proof of this result is the observation that if u is positive and harmonic on H, then $\mathcal{K}[u]$ is positive and harmonic on B, and hence is the Poisson integral of a positive measure on S. The restriction of this measure to $S \setminus \{\mathbf{S}\}$ gives rise to the measure μ, and the mass of this measure at $\mathbf{S}$ gives rise to the term cy.

Before coming to the proof of Theorem 7.26 proper, we need to understand how measures on S pull back, via the map Φ, to measures

on $\mathbf{R}^{n-1}$. For any positive $\nu \in M(S)$, we can define a positive measure $\nu \circ \Phi \in M(\mathbf{R}^{n-1})$ by setting $(\nu \circ \Phi)(E) = \nu(\Phi(E))$ for every Borel set $E \subset \mathbf{R}^{n-1}$. We then have the following "change of variables formula", valid for every positive Borel measurable function f on $S \setminus \{\mathbf{S}\}$:

7.27
$$\int_{S\setminus\{\mathbf{S}\}} f\,d\nu = \int_{\mathbf{R}^{n-1}} (f \circ \Phi)\,d(\nu \circ \Phi).$$

The last equation is easy to verify when f is a simple function on $S \setminus \{\mathbf{S}\}$; the full result follows from this by the monotone convergence theorem.

PROOF OF THEOREM 7.26: If u is positive and harmonic on H, then $\mathcal{K}[u]$ is positive and harmonic on B, and thus $\mathcal{K}[u] = P[\lambda]$ for some positive measure $\lambda \in M(S)$. Define $\nu \in M(S)$ by $d\nu = \chi_{S\setminus\{\mathbf{S}\}}\,d\lambda$. We then have

$$\mathcal{K}[u] = P[\nu] + \lambda(\{\mathbf{S}\})P(\cdot, \mathbf{S}).$$

By the linearity of $\mathcal{K}$,

$$u = \mathcal{K}[P[\nu]] + \lambda(\{\mathbf{S}\})\mathcal{K}[P(\cdot, \mathbf{S})].$$

From Theorem 7.22 it is easy to see that $\mathcal{K}[P(\cdot, \mathbf{S})]$ is a constant multiple of y on H. The proof will be completed by showing that $\mathcal{K}[P[\nu]] = P_H[\mu]$ for some positive Borel measure μ on $\mathbf{R}^{n-1}$.

Because $\nu(\{\mathbf{S}\}) = 0$,

$$P[\nu](z) = \int_{S\setminus\{\mathbf{S}\}} P(z, \zeta)\,d\nu(\zeta)$$

for all $z \in B$. Thus by 7.27,

$$\begin{aligned}\mathcal{K}[P[\nu]](z) &= \int_{S\setminus\{\mathbf{S}\}} 2^{(n-2)/2}|z - \mathbf{S}|^{2-n}P(\Phi(z), \zeta)\,d\nu(\zeta)\\ &= \int_{\mathbf{R}^{n-1}} 2^{(n-2)/2}|z - \mathbf{S}|^{2-n}P(\Phi(z), \Phi(t))\,d(\nu \circ \Phi)(t)\end{aligned}$$

for every $z \in H$. In the last integral we may multiply and divide by $\psi(t)$, where $\psi(t) = 2^{(n-2)/2}c_n(1 + |t|^2)^{-n/2}$. With $d\mu = (1/\psi)\,d(\nu \circ \Phi)$, we then have $\mathcal{K}[P[\nu]] = P_H[\mu]$, as desired. ∎

Nontangential Limits

We now look briefly at the Fatou Theorem for the Poisson integrals discussed in this chapter. Rather than tediously verifying that the maximal function arguments of the last chapter carry through to the present setting, we use the modified Kelvin transform $\mathcal{K}$ to transfer the Fatou Theorem from B to H.

The notion of a nontangential limit for a function on H was defined in Chapter 2; the analogous definition for a function on B was given in Chapter 6. We leave it to the reader to verify the following assertion, which follows from the conformality of the map Φ: a function u on H has a nontangential limit at $t \in \mathbf{R}^{n-1}$ if and only if $\mathcal{K}[u]$ has a nontangential limit (within B) at $\Phi(t)$.

Another observation that we leave to the reader is that the map Φ preserves sets of measure zero. More precisely, a Borel set $E \subset \mathbf{R}^{n-1}$ has Lebesgue measure 0 if and only if $\Phi(E)$ has σ-measure 0 on S; this follows easily from the smoothness of Φ.

In this chapter, the term "almost everywhere" will refer to Lebesgue measure on $\mathbf{R}^{n-1}$. Putting the last two observations together, we see that a function u on H has nontangential limits almost everywhere on $\mathbf{R}^{n-1}$ if and only if $\mathcal{K}[u]$ has nontangential limits σ-almost everywhere on S.

The next result is the Fatou theorem for Poisson integrals of functions in $L^p(\mathbf{R}^{n-1})$.

7.28 Theorem: *Let $p \in [1, \infty]$. If $f \in L^p(\mathbf{R}^{n-1})$, then $P_H[f]$ has nontangential limit $f(x)$ at almost every $x \in \mathbf{R}^{n-1}$.*

PROOF: Because every real-valued function in $L^p(\mathbf{R}^{n-1})$ is the difference of two positive functions in $L^p(\mathbf{R}^{n-1})$, we may assume that $f \geq 0$. The function $u = P_H[f]$ is then positive and harmonic on H, and thus $\mathcal{K}[u]$ is positive and harmonic on B. By 6.15 and 6.44, $\mathcal{K}[u]$ has nontangential limits σ-almost everywhere on S. As observed earlier, this implies that u has a nontangential limit $g(x)$ for almost every $x \in \mathbf{R}^{n-1}$.

We need to verify that $f = g$ almost everywhere. For $p < \infty$, Theorem 7.8 asserts that $\|u_y - f\|_p \to 0$ as $y \to 0$; thus some subsequence (u_{y_k}) converges to f pointwise almost everywhere on $\mathbf{R}^{n-1}$, and hence $f = g$. For $p = \infty$, Theorem 7.10(b) shows that $u_y \to f$ weak*

in $L^\infty(\mathbf{R}^{n-1})$ as $y \to 0$. But we also have $u_y \to g$ weak* in $L^\infty(\mathbf{R}^{n-1})$ by the dominated convergence theorem, and so we conclude $f = g$. ■

The theorem above shows, by Theorem 7.17, that if $u \in h^p(H)$ and $p \in (1, \infty]$, then u has nontangential limits almost everywhere on $\mathbf{R}^{n-1}$. The next theorem gives us the same result for $h^1(H)$ as a corollary.

7.29 Theorem: *Suppose $\mu \in M(\mathbf{R}^{n-1})$ is singular with respect to Lebesgue measure. Then $P_H[\mu]$ has nontangential limit 0 almost everywhere on $\mathbf{R}^{n-1}$.*

PROOF: We may assume that μ is positive. By analogy with 7.27, we define $\mu \circ \Phi \in M(S)$ by setting $(\mu \circ \Phi)(E) = \mu(\Phi(E \setminus \{\mathbf{S}\}))$ for every Borel set $E \subset S$. We then have

$$\mathcal{K}[P[\mu \circ \Phi]] = P_H[\nu],$$

where $d\nu = (1/\psi)\,d\mu$ and ψ is as in the proof of 7.26. Because μ is singular with respect to Lebesgue measure on $\mathbf{R}^{n-1}$, $\mu \circ \Phi$ is singular with respect to σ. By 6.42, $P[\mu \circ \Phi]$ has nontangential limit 0 almost everywhere on S. The equation above tells us that $P_H[\nu]$ has nontangential limit 0 almost everywhere on $\mathbf{R}^{n-1}$. From this we easily deduce that $P_H[\mu]$ has nontangential limit 0 almost everywhere on $\mathbf{R}^{n-1}$. ■

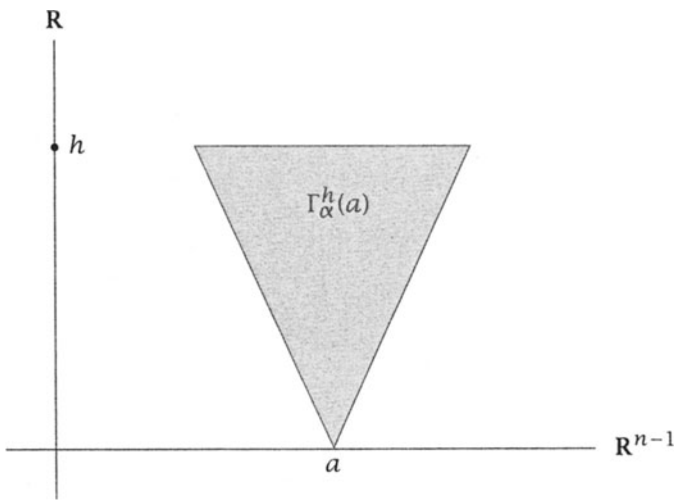
The Local Fatou Theorem

The Fatou Theorems obtained so far in this book apply to Poisson integrals of functions or measures. In this section we prove a different kind of Fatou theorem—one that applies to arbitrary harmonic functions on H satisfying a certain local boundedness condition.

We will need to consider truncations of the cones $\Gamma_\alpha(a)$ defined in Chapter 2. Thus, for any $h > 0$, we define

$$\Gamma_\alpha^h(a) = \{(x, y) \in H : |x - a| < \alpha y \text{ and } y < h\}.$$

A function u on H is said to be *nontangentially bounded* at $a \in \mathbf{R}^{n-1}$ if u is bounded on some $\Gamma_\alpha^h(a)$. Note that if u is continuous on H, then u is nontangentially bounded at a if and only if u is bounded on $\Gamma_\alpha^1(a)$ for some $\alpha > 0$. We can now state the main result of this section.



The truncated cone $\Gamma_\alpha^h(a)$.

7.30 Local Fatou Theorem: *Suppose that u is harmonic on H and $E \subset \mathbf{R}^{n-1}$ is the set of points at which u is nontangentially bounded. Then u has a nontangential limit at almost every point of E.*

A remarkable feature of this theorem should be emphasized. For each $a \in E$, we are only assuming that u is bounded in some $\Gamma_\alpha^h(a)$; in particular, α can depend on a. Nevertheless, the theorem asserts the existence of a set of full measure $F \subset E$ such that u has a limit in $\Gamma_\alpha(a)$ for every $a \in F$ and every $\alpha > 0$.

The following lemma will be important in proving the Local Fatou Theorem. Figure 7.32 may be helpful in picturing the geometry of the region Ω mentioned in the next three lemmas.

7.31 Lemma: *Suppose $E \subset \mathbf{R}^{n-1}$ is Borel measurable, $\alpha > 0$, and*

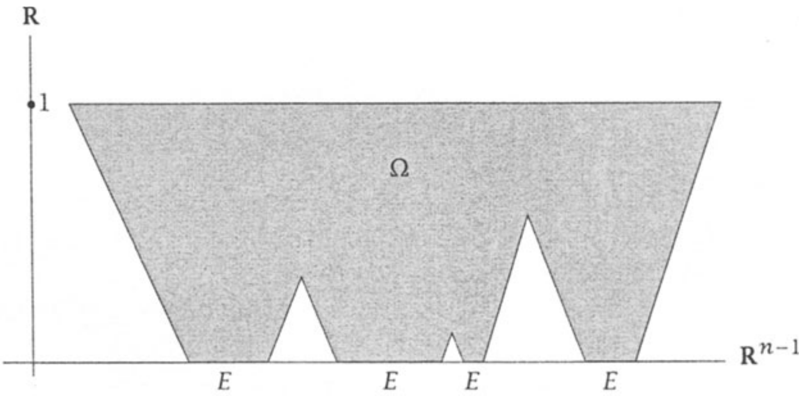
$$\Omega = \bigcup_{a \in E} \Gamma_\alpha^1(a).$$

Then there exists a positive harmonic function v on H such that $v \geq 1$ on $(\partial\Omega) \cap H$ and such that v has nontangential limit 0 almost everywhere on E.

PROOF: Define a positive harmonic function w on H by

$$w(x,y) = P_H[\chi_{E^c}](x,y) + y,$$

where χ_{E^c} denotes the characteristic function of E^c, the complement of E in $\mathbf{R}^{n-1}$. By Theorem 7.28, w has nontangential limit 0 almost everywhere on E.



7.32 $\Omega = \bigcup_{a \in E} \Gamma_\alpha^1(a).$

We wish to show that w is bounded away from 0 on $(\partial\Omega) \cap H$. Because $w(x,1) \geq 1$, we have $w \geq 1$ on the "top" of $\partial\Omega$. Next, observe that (x,y) belongs to $\Gamma_\alpha(a)$ if and only if $a \in B(x,\alpha y)$ (where $B(x,\alpha y)$ denotes the ball in $\mathbf{R}^{n-1}$ with center x and radius αy). So if $(x,y) \in \partial\Omega$ and $0 < y < 1$, then $(x,y) \notin \Gamma_\alpha(a)$ for all $a \in E$ (otherwise $(x,y) \in \Omega$), giving $B(x,\alpha y) \subset E^c$. Therefore

$$\begin{aligned} P_H[\chi_{E^c}](x,y) &= \int_{E^c} \frac{c_n y}{(|x-t|^2 + y^2)^{n/2}}\, dt \\ &\geq \int_{B(x,\alpha y)} \frac{c_n y}{(|x-t|^2 + y^2)^{n/2}}\, dt \\ &= \int_{B(0,\alpha)} \frac{c_n}{(|t|^2 + 1)^{n/2}}\, dt. \end{aligned}$$

Denoting the last expression by c_α (a constant less than 1 that depends only on α and n), we see that if $v = w/c_\alpha$, then v satisfies the conclusion of the lemma. ■

The crux of the proof of 7.30 is the following weaker version of the Local Fatou Theorem.

7.33 Lemma: *Let $E \subset \mathbf{R}^{n-1}$ be Borel measurable, let $\alpha > 0$, and let*

$$\Omega = \bigcup_{a \in E} \Gamma_\alpha^1(a).$$

Suppose u is harmonic on H and bounded on Ω. Then for almost every $a \in E$, the limit of $u(z)$ exists as $z \to a$ within $\Gamma_\alpha(a)$.

PROOF: Because every Borel set can be written as a countable union of bounded Borel sets, we may assume E is bounded. We may also assume that u is real valued.

Because u is continuous on H and E is bounded, we may assume that $|u| \le 1$ on the open set

$$\Omega' = \bigcup_{a \in E} \Gamma_\alpha^2(a).$$

Choose a sequence (y_k) in the interval $(0, 1)$ such that $y_k \to 0$, and set $E_k = [\Omega - (0, y_k)] \cap \mathbf{R}^{n-1}$. Each E_k is an open subset of $\mathbf{R}^{n-1}$ that contains E. (At this point we suggest the reader start drawing some pictures.)

For $x \in \mathbf{R}^{n-1}$, define

$$f_k(x) = \chi_{E_k}(x) u(x, y_k).$$

Because $(x, y_k) \in \Omega$ if and only if $x \in E_k$, we have $|f_k| \le 1$ on $\mathbf{R}^{n-1}$ for every k. The sequence (f_k), being norm-bounded in $L^\infty(\mathbf{R}^{n-1})$, has a subsequence, which we still denote by (f_k), that converges weak* to some $f \in L^\infty(\mathbf{R}^{n-1})$.

Now each f_k is continuous on E_k (because E_k is open), and thus $P_H[f_k]$ extends continuously to $H \cup E_k$ (see Exercise 17(a) of this chapter). The function u_k given by

$$u_k(x, y) = P_H[f_k](x, y) - u(x, y + y_k)$$

is thus harmonic on H and extends continuously to $H \cup E_k$, with $u_k = 0$ on E_k. In particular, u_k is continuous on $\overline{\Omega}$ with $u_k = 0$ on E. Furthermore, because $\Omega + (0, y_k) \subset \Omega'$, we have $|u_k| \leq 2$ on $\overline{\Omega}$.

Now let v denote the function of Lemma 7.31 with respect to Ω. Then $\liminf_{z \to \partial\Omega}(2v - u_k)(z) \geq 0$. By the minimum principle (1.10), $2v - u_k \geq 0$ on Ω. Letting $k \to \infty$, we then see that $2v - (P_H[f] - u) \geq 0$ on Ω. Because this argument applies as well to $2v + u_k$, we conclude that $|P_H[f] - u| \leq 2v$ on Ω.

By Theorem 7.28, $P_H[f]$ has nontangential limits almost everywhere on $\mathbf{R}^{n-1}$, while Lemma 7.31 asserts v has nontangential limits 0 almost everywhere on E. From this and the last inequality, the desired limits for u follow. ∎

Recall that if $E \subset \mathbf{R}^{n-1}$ is Borel measurable, then a point $a \in E$ is said to be a *point of density* of E provided

$$\lim_{r \to 0} \frac{V_{n-1}(B(a,r) \cap E)}{V_{n-1}(B(a,r))} = 1.$$

By the Lebesgue Differentiation Theorem ([15], Theorem 7.7), almost every point of E is a point of density of E.

Points of density of E are where we can expect the cones defining Ω in Lemma 7.33 to "pile up"; this will allow us to pass from 7.33 to the stronger assertion in 7.30.

7.34 Lemma: *Suppose $E \subset \mathbf{R}^{n-1}$ is Borel measurable, $\alpha > 0$, and*

$$\Omega = \bigcup_{a \in E} \Gamma_\alpha^1(a).$$

Suppose u is continuous on H and bounded on Ω. If a is a point of density of E, then u is bounded in $\Gamma_\beta^1(a)$ for every $\beta > 0$.

PROOF: Let a be a point of density of E, and let $\beta > 0$. It suffices to show that $\Gamma_\beta^h(a) \subset \Omega$ for some $h > 0$.

Choose $\delta > 0$ such that

$$\text{7.35} \qquad \frac{V_{n-1}(B(a,r) \cap E)}{V_{n-1}(B(a,r))} > 1 - \left(\frac{\alpha}{\alpha + \beta}\right)^{n-1}$$

whenever $r < \delta$; we may assume $\delta/(\alpha + \beta) < 1$. Set $h = \delta/(\alpha + \beta)$, and let $(x, y) \in \Gamma_\beta^h(a)$. Then $B(x, \alpha y) \subset B(a, (\alpha + \beta)y)$. This implies

$B(x, \alpha y) \cap E$ is nonempty; otherwise we violate 7.35 (take $r = (\alpha+\beta)y$). Choosing any $b \in B(x, \alpha y) \cap E$, we have $(x, y) \in \Gamma_\alpha^1(b)$, and thus $\Gamma_\beta^h(a) \subset \Omega$, as desired. ■

PROOF OF THEOREM 7.30: We are assuming u is harmonic on H and E is the set of points in $\mathbf{R}^{n-1}$ at which u is nontangentially bounded. For $k = 1, 2, \ldots$, set $E_k = \{a \in \mathbf{R}^{n-1} : |u| \le k \text{ on } \Gamma_{1/k}^1(a)\}$. Then each E_k is a closed subset of $\mathbf{R}^{n-1}$, and $E = \bigcup E_k$ (incidentally proving that the set E is Borel measurable). Applying Lemma 7.34 to each E_k, and recalling that the points of density of E_k form a set of full measure in E_k, we see that there is a set of full measure $F \subset E$ such that u is bounded on $\Gamma_\alpha^1(a)$ for every $a \in F$ and every $\alpha > 0$. For each positive integer k, we can write F as $F = \bigcup F_j$, where $F_j = \{a \in F : |u| \le j \text{ on } \Gamma_k^1(a)\}$. Lemma 7.33, applied to F_j, now shows that u has nontangential limits almost everywhere on E, as desired. ■

Exercises

1. Assume $n = 2$. For each $t \in \mathbf{R}$, find a holomorphic function g_t on H such that $P_H(\cdot, t) = \operatorname{Re} g_t$.

2. In Chapter 1, we calculated $P(x, \zeta)$ as a normal derivative on ∂B of an appropriate modification of $|x - \zeta|^{2-n}$ ($n > 2$). Using an appropriate modification of $|z-t|^{2-n}$, find a function whose normal derivative on ∂H is $P_H(z, t)$.

3. Let $\mu \in M(\mathbf{R}^{n-1})$ and let $u = P_H[\mu]$. Prove that

$$\int_{B_{n-1}} u_y(x)\, dx \to \mu(B_{n-1}) + \frac{\mu(\partial B_{n-1})}{2}$$

as $y \to 0$.

4. Let $p \in [1, \infty]$ and assume $u \in h^p(H)$. Show that if the functions u_y converge uniformly on $\mathbf{R}^{n-1}$ as $y \to 0$, then u extends to a bounded uniformly continuous function on $\overline{H}$.

5. For $\zeta \in S$, show that

$$\Phi(\zeta) = \frac{(\zeta_1, \zeta_2, \dots, \zeta_{n-1}, 0)}{1 + \zeta_n}.$$

6. For $(x, y) \in \mathbf{R}^n \setminus \{\mathbf{S}\}$, show that

$$1 - |\Phi(x, y)|^2 = \frac{4y}{|x|^2 + (y+1)^2}.$$

7. Show that if $n = 2$, then

$$\Phi(z) = \frac{1 - i\overline{z}}{\overline{z} - i}$$

for every $z \in \mathbf{C} \setminus \{-i\}$.

8. Suppose $\zeta \in S$ and $f \in C(S \setminus \{\zeta\})$. Prove that there exists $u \in C(\overline{B} \setminus \{\zeta\})$ such that u is harmonic on B and $u|_{S \setminus \{\zeta\}} = f$.

9. Prove that

$$\int_S f\,d\sigma = c_n 2^{n-2} \int_{\mathbf{R}^{n-1}} f(\Phi(t))(1+|t|^2)^{1-n}\,dt$$

for every positive Borel measurable function f on S. (*Hint:* Because $\Phi: S\setminus\{\mathbf{S}\} \to \mathbf{R}^{n-1}$ is smooth, there exists a smooth function w on $\mathbf{R}^{n-1}$ such that $d(\sigma\circ\Phi) = w\,dt$. To find w, apply 7.27 with $\nu = \sigma$.)

10. Using the result of the last exercise, show that

$$\int_{\mathbf{R}^{n-1}} f(t)\,dt = \frac{2}{c_n}\int_S f(\Phi(\zeta))(\zeta_n+1)^{1-n}\,d\sigma(\zeta)$$

for every positive Borel measurable function f on $\mathbf{R}^{n-1}$.

11. (a) Let μ be a positive Borel measure on $\mathbf{R}^{n-1}$ that satisfies 7.25, and set $u = P_H[\mu]$. Show that $\lim_{y\to\infty} u(0,y)/y = 0$.

    (b) Let u be a positive harmonic function on H. Show that $\liminf_{y\to 0} u(0,y)/y > 0$.

12. Show that if u is positive and harmonic on H, then the decomposition $u(x,y) = P_H[\mu](x,y) + cy$ of Theorem 7.26 holds for a unique positive Borel measure μ on $\mathbf{R}^{n-1}$ and a unique nonnegative constant c.

13. Let μ be a positive Borel measure on $\mathbf{R}^{n-1}$ that satisfies 7.25, and set $u = P_H[\mu]$. Prove that

$$\lim_{y\to 0}\int_{\mathbf{R}^{n-1}} \varphi(t)u_y(t)\,dt = \int_{\mathbf{R}^{n-1}} \varphi(t)\,d\mu(t)$$

for every continuous function φ on $\mathbf{R}^{n-1}$ with compact support.

14. Prove that $\mathcal{K}[h^p(H)] \subset h^1(B)$ for every $p \in [1,\infty]$. (*Hint:* Exercise 9 in Chapter 6 may be helpful here.)

15. Let $p \in [1,\infty]$ and let $f \in L^p(\mathbf{R}^{n-1})$. Show that $\mathcal{K}[f] \in L^1(S)$.

16. Let $p \in [1,\infty]$ and let $f \in L^p(\mathbf{R}^{n-1})$. Show that

$$\mathcal{K}\Big[P[\mathcal{K}[f]]\Big](z) = P_H[f](z)$$

for every $z \in H$.

17. Assume that f is measurable on $\mathbf{R}^{n-1}$ and that

$$\int_{\mathbf{R}^{n-1}} |f(t)|(1 + |t|^2)^{-n/2}\, dt < \infty.$$

(a) Show that if f is continuous at a, then $P_H[f] \to f(a)$ as $z \to a$ within H.

(b) Show that $P_H[f]$ tends nontangentially to f almost everywhere on $\mathbf{R}^{n-1}$. (*Hint:* Let g denote f times the characteristic function of some large ball. Apply Theorem 7.28 to $P_H[g]$; apply part (a) to $P_H[f - g]$.)

18. Let μ be a positive Borel measure on $\mathbf{R}^{n-1}$ that satisfies 7.25. Show that if μ is singular with respect to Lebesgue measure, then $P_H[\mu]$ has nontangential limit 0 at almost every point of $\mathbf{R}^{n-1}$.

CHAPTER 8

Harmonic Bergman Spaces

Throughout this chapter, p denotes a number satisfying $1 \le p < \infty$. The *Bergman space* $b^p(\Omega)$ is the set of harmonic functions u on Ω such that

$$\|u\|_{b^p} = \left(\int_\Omega |u|^p \, dV \right)^{1/p} < \infty.$$

We often view $b^p(\Omega)$ as a subspace of $L^p(\Omega, dV)$. The spaces $b^p(\Omega)$ are named in honor of Stefan Bergman, who studied analogous spaces of holomorphic functions.



Stefan Bergman (1895–1977), whose book [5] popularized the study of spaces of holomorphic functions belonging to L^p with respect to volume measure.

Reproducing Kernels

For fixed $x \in \Omega$, the map $u \mapsto u(x)$ is a linear functional on $b^p(\Omega)$; we refer to this map as *point evaluation* at x. The following proposition shows that point evaluation is continuous on $b^p(\Omega)$.

8.1 Proposition: *Suppose $x \in \Omega$. Then*

$$|u(x)| \le \frac{1}{V(B)^{1/p} d(x, \partial\Omega)^{n/p}} \|u\|_{b^p}$$

for every $u \in b^p(\Omega)$.

PROOF: Let r be a positive number with $r < d(x, \partial\Omega)$, and apply the volume version of the mean-value property to u on $B(x,r)$. After taking absolute values, Jensen's inequality gives

$$|u(x)|^p \le \frac{1}{V(B(x,r))} \int_{B(x,r)} |u|^p \, dV \le \frac{1}{r^n V(B)} \|u\|_{b^p}^p.$$

The desired inequality is now obtained by taking p^{th} roots and letting $r \to d(x, \partial\Omega)$. ■

The next result shows that point evaluation of every partial derivative is also continuous of $b^p(\Omega)$.

8.2 Corollary: *For every multi-index α there exists a constant C_α such that*

$$|D^\alpha u(x)| \le \frac{C_\alpha}{d(x, \partial\Omega)^{|\alpha| + n/p}} \|u\|_{b^p}$$

for all $x \in \Omega$ and every $u \in b^p(\Omega)$.

PROOF: Apply 8.1 and Cauchy's Estimates (2.4) to u on the ball of radius $d(x, \partial\Omega)/2$ centered at x. ■

The next proposition implies that $b^p(\Omega)$ is a Banach space.

8.3 Proposition: *The Bergman space $b^p(\Omega)$ is a closed subspace of $L^p(\Omega, dV)$.*

PROOF: Suppose $u_j \to u$ in $L^p(\Omega, dV)$, where (u_j) is a sequence in $b^p(\Omega)$ and $u \in L^p(\Omega, dV)$. We must show that, after appropriate modification on a set of measure zero, u is harmonic on Ω.

Let $K \subset \Omega$ be compact. By Proposition 8.1, there is a constant $C < \infty$ such that

$$|u_j(x) - u_k(x)| \le C\|u_j - u_k\|_{b^p}$$

for all $x \in K$ and all j, k. Because (u_j) is a Cauchy sequence in $b^p(\Omega)$, the inequality above implies that (u_j) is a Cauchy sequence in $C(K)$. Hence (u_j) converges uniformly on K.

Thus (u_j) converges uniformly on compact subsets of Ω to a function v that is harmonic on Ω (Theorem 1.23).

Because $u_j \to u$ in $L^p(\Omega, dV)$, some subsequence of (u_j) converges to u pointwise almost everywhere on Ω. It follows that $u = v$ almost everywhere on Ω, and thus $u \in b^p(\Omega)$, as desired. ∎

Taking $p = 2$, we see that the last proposition shows that $b^2(\Omega)$ is a Hilbert space with inner product

$$\langle u, v\rangle = \int_\Omega u\overline{v}\, dV.$$

For each $x \in \Omega$, the map $u \mapsto u(x)$ is a bounded linear functional on the Hilbert space $b^2(\Omega)$ (by Proposition 8.1). Thus there exists a unique function $R_\Omega(x, \cdot) \in b^2(\Omega)$ such that

$$u(x) = \int_\Omega u(y)\overline{R_\Omega(x, y)}\, dV(y)$$

for every $u \in b^2(\Omega)$. The function R_Ω, which can be viewed as a function on $\Omega \times \Omega$, is called the *reproducing kernel* of Ω.

The basic properties of R_Ω given below are analogous to properties of the zonal harmonics we studied in Chapter 5 (even the proofs are the same).

8.4 Proposition: *The reproducing kernel of Ω has the following properties:*

(a) *R_Ω is real valued.*

(b) *If (u_m) is an orthonormal basis of $b^2(\Omega)$, then*

$$R_\Omega(x,y) = \sum_{m=1}^{\infty} \overline{u_m(x)} u_m(y)$$

for all $x, y \in \Omega$.

(c) *$R_\Omega(x,y) = R_\Omega(y,x)$ for all $x, y \in \Omega$.*

(d) *$\|R_\Omega(x,\cdot)\|_{b^2} = \sqrt{R_\Omega(x,x)}$ for all $x \in \Omega$.*

PROOF: To prove (a), suppose that $u \in b^2(\Omega)$ is real valued and $x \in \Omega$. Then

$$\begin{aligned} 0 &= \operatorname{Im} u(x) \\ &= \operatorname{Im} \int_\Omega u(y) \overline{R_\Omega(x,y)}\, dV(y) \\ &= -\int_\Omega u(y) \operatorname{Im} R_\Omega(x,y)\, dV(y). \end{aligned}$$

Take $u = \operatorname{Im} R_\Omega(x,\cdot)$, obtaining

$$\int_\Omega (\operatorname{Im} R_\Omega(x,y))^2\, dV(y) = 0,$$

which implies $\operatorname{Im} R_\Omega \equiv 0$. We conclude that each R_Ω is real valued, as desired.

To prove (b), let (u_m) be any orthonormal basis of $b^2(\Omega)$. (Recall that $L^2(\Omega, dV)$, and hence $b^2(\Omega)$, is separable.) By standard Hilbert space theory,

$$\begin{aligned} R_\Omega(x,\cdot) &= \sum_{m=1}^{\infty} \langle R_\Omega(x,\cdot), u_m \rangle u_m \\ &= \sum_{m=1}^{\infty} \overline{u_m(x)} u_m \end{aligned}$$

for each $x \in \Omega$, where the infinite sums converge in norm in $b^2(\Omega)$. Since point evaluation is a continuous linear functional on $b^2(\Omega)$, the equation above shows that the conclusion of (b) holds.

To prove (c), note that (b) shows that $\overline{R_\Omega(x,y)} = R_\Omega(y,x)$, while (a) shows that $\overline{R_\Omega(x,y)} = R_\Omega(x,y)$ for all $x, y \in \Omega$. Putting these two equations together gives (c).

To prove (d), let $x \in \Omega$. Then

$$\begin{aligned}\|R_\Omega(x,\cdot)\|_{b^2}^2 &= \langle R_\Omega(x,\cdot), R_\Omega(x,\cdot)\rangle \\ &= R_\Omega(x,x),\end{aligned}$$

where the second equality follows from the reproducing property of $R_\Omega(x,\cdot)$. Taking square roots gives (d). ∎

Because $b^2(\Omega)$ is a closed subspace of the Hilbert space $L^2(\Omega, dV)$, there is a unique orthogonal projection of $L^2(\Omega, dV)$ onto $b^2(\Omega)$. This self-adjoint projection is called the *Bergman projection* on Ω; we denote it by Q_Ω. The next proposition establishes the connection between the Bergman projection and the reproducing kernel.

8.5 Proposition: *If $x \in \Omega$, then*

$$Q_\Omega[u](x) = \int_\Omega u(y) R_\Omega(x,y)\, dV(y)$$

for all $u \in L^2(\Omega, dV)$.

PROOF: Let $x \in \Omega$ and $u \in L^2(\Omega, dV)$. Then

$$\begin{aligned}Q_\Omega[u](x) &= \langle Q_\Omega[u], R_\Omega(x,\cdot)\rangle \\ &= \langle u, R_\Omega(x,\cdot)\rangle \\ &= \int_\Omega u(y) R_\Omega(x,y)\, dV(y),\end{aligned}$$

where the first equality above follows from the reproducing property of $R_\Omega(x,\cdot)$, the second equality holds because Q_Ω is a self-adjoint projection onto a subspace containing $R_\Omega(x,\cdot)$, and the third equality follows from the definition of the inner product and Proposition 8.4(a). ∎

In the next section, we will find a formula for computing $Q_B[p]$ when p is a polynomial; see 8.14 and 8.15.

The Reproducing Kernel of the Ball

In this section we will find an explicit formula for the reproducing kernel of the unit ball. We begin by looking at the space $\mathcal{H}_m(\mathbf{R}^n)$, which consists of the harmonic polynomials on $\mathbf{R}^n$ that are homogeneous of degree m. Recall that the zonal harmonics introduced in Chapter 5 are reproducing kernels for $\mathcal{H}_m(\mathbf{R}^n)$. Thus if $p \in \mathcal{H}_m(\mathbf{R}^n)$, then

8.6
$$p(x) = \int_S p(\zeta) Z_m(x,\zeta)\, d\sigma(\zeta)$$

for each $x \in \mathbf{R}^n$ (by 5.30). By using polar coordinates, we will obtain an analogue of 8.6 involving integration over B instead of S.

First we extend the zonal harmonic Z_m to a function on $\mathbf{R}^n \times \mathbf{R}^n$. We do this by making Z_m homogeneous in the second variable as well as in the first; in other words, we set

8.7
$$Z_m(x,y) = |x|^m |y|^m Z_m(x/|x|, y/|y|).$$

(If either x or y is 0, we define $Z_m(x,y)$ to be 0 when $m > 0$; when $m = 0$, we define Z_0 to be identically 1.) With this extended definition, $Z_m(x,\cdot) \in \mathcal{H}_m(\mathbf{R}^n)$ for each $x \in \mathbf{R}^n$; also, $Z_m(x,y) = Z_m(y,x)$ for all $x, y \in \mathbf{R}^n$.

We now derive the analogue of 8.6 for integration over B. For every $p \in \mathcal{H}_m(\mathbf{R}^n)$, we have

$$\begin{aligned}
&\int_B p(y) Z_m(x,y)\, dV(y)\\
&\qquad = nV(B) \int_0^1 r^{n-1} \int_S p(r\zeta) Z_m(x, r\zeta)\, d\sigma(\zeta)\, dr\\
&\qquad = nV(B) \int_0^1 r^{n+2m-1} \int_S p(\zeta) Z_m(x,\zeta)\, d\sigma(\zeta)\, dr\\
&\qquad = nV(B) p(x) \int_0^1 r^{n+2m-1}\, dr\\
&\qquad = \frac{nV(B)}{n+2m} p(x)
\end{aligned}$$

for each $x \in \mathbf{R}^n$. In other words, $p(x)$ equals the inner product of p with $(n+2m)Z_m(x,\cdot)/(nV(B))$ for every $p \in \mathcal{H}_m(\mathbf{R}^n)$.

Now, $\mathcal{H}_k(\mathbf{R}^n)$ is orthogonal to $\mathcal{H}_m(\mathbf{R}^n)$ in $b^2(B)$ if $k \neq m$, as can be verified using Proposition 5.9 and polar coordinates (1.5). Thus if p is a harmonic polynomial of degree M, then $p(x)$ equals the inner product of p with $\sum_{m=0}^{M}(n+2m)Z_m(x,\cdot)/(nV(B))$. Taking $M = \infty$ in the last sum gives us a good candidate for the reproducing kernel of the ball; Theorem 8.9 will show that this is the right guess. The following lemma will be useful in proving this theorem.

8.8 Lemma: *The set of harmonic polynomials is dense in $b^2(B)$.*

PROOF: First note that if $u \in L^2(B, dV)$, then $u_r \to u$ in $L^2(B, dV)$ as $r \to 1$. (For $u \in C(\overline{B})$, use uniform continuity; the general result follows because $C(\overline{B})$ is dense in $L^2(B, dV)$.) Thus any $u \in b^2(B)$ can be approximated in $b^2(B)$ by functions harmonic on $\overline{B}$. But by 5.34, every function harmonic on $\overline{B}$ can be approximated uniformly on $\overline{B}$, and hence in $L^2(B, dV)$, by harmonic polynomials. ■

Now we can express the reproducing kernel of the ball as an infinite linear combination of zonal harmonics. We will use the theorem below to derive an explicit formula for R_B.

8.9 Theorem: *If $x, y \in B$ then*

8.10
$$R_B(x,y) = \frac{1}{nV(B)} \sum_{m=0}^{\infty} (n+2m) Z_m(x,y).$$

The series converges absolutely and uniformly on $K \times B$ for every compact $K \subset B$.

PROOF: For $x, y \in B \setminus \{0\}$ we have

$$\begin{aligned} |Z_m(x,y)| &= |x|^m |y|^m |Z_m(x/|x|, y/|y|)| \\ &\leq |x|^m |y|^m \dim \mathcal{H}_m(\mathbf{R}^n), \end{aligned}$$

where the inequality comes from Proposition 5.27(e). Now Exercise 10 in Chapter 5 shows that the infinite series in 8.10 has the convergence properties claimed in the theorem. Thus if $F(x,y)$ denotes the right side of 8.10, then $F(x,\cdot)$ is a bounded harmonic function on B for each $x \in B$. In particular, $F(x,\cdot) \in b^2(B)$ for each $x \in B$.

Now fix $x \in B$. The discussion before the statement of Lemma 8.8 shows that $p(x) = \langle p, F(x,\cdot)\rangle$ whenever p is a harmonic polynomial. Because point evaluation is continuous on $b^2(B)$ and harmonic polynomials are dense in $b^2(B)$, we have $u(x) = \langle u, F(x,\cdot)\rangle$ for all $u \in b^2(B)$. Hence F is the reproducing kernel of the ball. ∎

Our next goal is to evaluate explicitly the infinite sum in 8.10. Before doing so, note that a natural guess about how to find a formula for the reproducing kernel would be to find an orthonormal basis of $b^2(B)$ and then try to evaluate the infinite sum in Proposition 8.4(b). This approach is feasible when $n = 2$ (see Exercise 14 of this chapter). However, there appears to be no canonical choice for an orthonormal basis of $b^2(B)$ when $n > 2$.

Recall (see 6.21) that the extended Poisson kernel is defined by

8.11
$$P(x,y) = \frac{1 - |x|^2|y|^2}{(1 - 2x \cdot y + |x|^2|y|^2)^{n/2}}$$

for all $x, y \in \mathbf{R}^n \times \mathbf{R}^n$ for which the denominator above is not 0. Following 6.21, we noted some properties of the extended Poisson kernel: $P(x,y) = P(y,x) = P(|x|y, x/|x|)$, and for x fixed, $P(x,\cdot)$ is a harmonic function.

The key connection between the extended Poisson kernel and R_B is the formula for the Poisson kernel given by Theorem 5.33, which states that

$$P(x,\zeta) = \sum_{m=0}^{\infty} Z_m(x,\zeta)$$

for $x \in B$ and $\zeta \in S$. For $x, y \in B$, this implies that

$$\begin{aligned}\sum_{m=0}^{\infty} Z_m(x,y) &= \sum_{m=0}^{\infty} Z_m(|y|x, y/|y|)\\ &= P(|y|x, y/|y|)\\ &= P(x,y).\end{aligned}$$

Returning to 8.10, observe that

$$\sum_{m=0}^{\infty} 2mZ_m(x,y) = \sum_{m=0}^{\infty} \frac{d}{dt} t^{2m} Z_m(x,y)|_{t=1}$$
$$= \frac{d}{dt}\Big(\sum_{m=0}^{\infty} t^{2m} Z_m(x,y)\Big)|_{t=1}$$
$$= \frac{d}{dt}\Big(\sum_{m=0}^{\infty} Z_m(tx,ty)\Big)|_{t=1}$$
$$= \frac{d}{dt} P(tx,ty)|_{t=1}.$$

Thus 8.10 implies the beautiful equation

8.12
$$R_B(x,y) = \frac{nP(x,y) + \frac{d}{dt}P(tx,ty)|_{t=1}}{nV(B)}.$$

This simple representation gives us a formula in closed form for the reproducing kernel R_B.

8.13 Theorem: *Let $x, y \in B$. Then*

$$R_B(x,y) = \frac{(n-4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{nV(B)(1 - 2x \cdot y + |x|^2|y|^2)^{1+n/2}}.$$

PROOF: Compute using 8.12 and 8.11. ∎

The next result gives a formula for the Bergman projection on the unit ball. It should be compared to Theorem 5.1 and Proposition 5.31.

8.14 Theorem: *Let p be a polynomial on $\mathbf{R}^n$ of degree m. Then $Q_B[p]$ is a polynomial of degree at most m. Moreover,*

$$Q_B[p](x) = \frac{1}{nV(B)} \sum_{k=0}^{m} (n+2k) \int_B p(y) Z_k(x,y)\, dV(y)$$

for every $x \in B$.

PROOF: Fix $x \in B$. For each $r \in (0,1)$, the function p_r is a polynomial of degree m. Thus by Proposition 5.9 we have

$$\int_S p(r\zeta)Z_k(x,\zeta)\,d\sigma(\zeta) = 0$$

for all $k > m$. Hence

$$\begin{aligned}\frac{\int_B p(y)Z_k(x,y)\,dV(y)}{nV(B)} &= \int_0^1 r^{n+k-1}\int_S p(r\zeta)Z_k(x,\zeta)\,d\sigma(\zeta)\,dr\\ &= 0\end{aligned}$$

for all $k > m$. Combining this result with 8.10 and Proposition 8.5 gives the desired equation. ■

Recall that $\mathcal{P}_m(\mathbf{R}^n)$ denotes the space of polynomials on $\mathbf{R}^n$ that are homogeneous of degree m. The following corollary shows how to compute the Bergman projection of a polynomial from its Poisson integral.

8.15 Corollary: *Suppose $p \in \mathcal{P}_m(\mathbf{R}^n)$ and that $\sum_{k=0}^m p_k$ is the solution to the Dirichlet problem for the ball with boundary data $p|_S$, where each $p_k \in \mathcal{H}_k(\mathbf{R}^n)$. Then*

$$Q_B[p] = \sum_{k=0}^{m} \frac{n+2k}{n+k+m} p_k.$$

PROOF: For $0 \le k \le m$ and $x \in B$, we have

$$\begin{aligned}\frac{\int_B p(y)Z_k(x,y)\,dV(y)}{nV(B)} &= \int_0^1 r^{n-1}\int_S p(r\zeta)Z_k(x,r\zeta)\,d\sigma(\zeta)\,dr\\ &= \int_0^1 r^{n+k+m-1}\int_S p(\zeta)Z_k(x,\zeta)\,d\sigma(\zeta)\,dr\\ &= \frac{p_k(x)}{n+k+m},\end{aligned}$$

where the last equality comes from Proposition 5.31. Combining the last equality with Theorem 8.14 now gives the desired result. ■

If p is a polynomial on $\mathbf{R}^n$, then the software described in Appendix B computes the Bergman projection $Q_B[p]$ by first computing

the Poisson integral $P[p]$ (using Theorem 5.21) and then uses the corollary above to compute $Q_B[p]$. For example, if $n = 6$ and $p(x) = x_1{}^4x_2$, then this software computes that

$$Q_B[p](x) = \frac{x_2 + |x|^4x_2}{40} - \frac{|x|^2x_1{}^2x_2}{2} + x_1{}^4x_2 - \frac{3|x|^2x_2}{56} + \frac{3x_1{}^2x_2}{7}.$$

Let us make an observation in passing. We have seen that if p is a polynomial on $\mathbf{R}^n$, then the Poisson integral $P[p]$ and the Bergman projection $Q_B[p]$ are both polynomials. Each can be thought of as the solution to a certain minimization problem. Specifically, $P[p]$ minimizes

$$\|p - u\|_{L^\infty(S)},$$

while $Q_B[p]$ minimizes

$$\|p - u\|_{L^2(B,dV)},$$

where both minimums are taken over all functions u harmonic on $\overline{B}$. Curiously, if p is a homogeneous polynomial, then the two harmonic approximations $P[p]$ and $Q_B[p]$ agree only if p is harmonic (see Exercise 19 in this chapter).

Examples in $b^p(B)$

Because the Poisson integral is a linear isometry of $L^p(S)$ onto $h^p(B)$ ($p > 1$) and of $M(S)$ onto $h^1(B)$ (Theorem 6.13), we easily see that $h^p(B) \neq h^q(B)$ whenever $p \neq q$. We now prove the analogous result for the Bergman spaces of B.

8.16 Proposition: *If $1 \le p < q < \infty$, then $b^q(B)$ is a proper subset of $b^p(B)$.*

PROOF: Suppose $1 \le p < q < \infty$. Because B has finite volume measure, clearly $b^q(B) \subset b^p(B)$. To prove that this inclusion is proper, consider the identity map from $b^q(B)$ into $b^p(B)$. This map is linear, one-to-one, and bounded (by Hölder's inequality). If this map were onto, then the inverse mapping would be continuous by the open mapping theorem, and so there would exist a constant $C < \infty$ such that

8.17 $$\|u\|_{b^q} \le C\|u\|_{b^p}$$

for all $u \in b^p(B)$.

We will show that 8.17 fails. For $m = 1, 2, \ldots$, choose a homogeneous harmonic polynomial u_m of degree m, with $u_m \not\equiv 0$. Integrating in polar coordinates (1.5), we find that

$$\|u_m\|_{b^p} = \Big(\int_S |u_m|^p \, d\sigma\Big)^{1/p} \Big(nV(B) \int_0^1 r^{pm+n-1} \, dr\Big)^{1/p},$$

a similar result holding for $\|u_m\|_{b^q}$. Because L^r-norms on S with respect to σ increase as r increases (Hölder's inequality), we have

$$\frac{\|u_m\|_{b^q}}{\|u_m\|_{b^p}} \ge \frac{\big(nV(B)/(qm+n)\big)^{1/q}}{\big(nV(B)/(pm+n)\big)^{1/p}}.$$

As $m \to \infty$, the expression on the right of the last inequality tends to ∞. Therefore 8.17 fails, proving that the identity map from $b^q(B)$ into $b^p(B)$ is not onto. Thus $b^q(B)$ is properly contained in $b^p(B)$, as desired. ∎

We turn now to some other properties of the Bergman spaces on the ball. First note that $h^p(B) \subset b^p(B)$ for all $p \in [1, \infty)$, as an easy integration in polar coordinates (1.5) shows. (In fact, $h^p(B) \subset b^q(B)$ for all $q < pn/(n-1)$; see Exercise 21 of this chapter.) However, each of the spaces $b^p(B)$ contains functions not belonging to any $h^q(B)$, as we show below. In fact, we will construct a function in every $b^p(B)$ that at every point of the unit sphere fails to have a radial limit; such a function cannot belong to any $h^q(B)$ by Corollary 6.44. We begin with a lemma that will be useful in this construction..

8.18 Lemma: *Let $f_m(\zeta) = e^{im\zeta_1}$ for $\zeta \in S$ and $m = 1, 2, \ldots$. Then $P[f_m] \to 0$ uniformly on compact subsets of B as $m \to \infty$.*

PROOF: Let $g \in C(S)$. Using the slice integration formula (A.5 in Appendix A), we see that $\int_S f_m g \, d\sigma$ equals a constant (depending only on n) times

$$\int_{-1}^1 (1-t^2)^{\frac{n-3}{2}} e^{imt} \int_{S_{n-1}} g(t, \sqrt{1-t^2}\,\zeta) \, d\sigma_{n-1}(\zeta) \, dt,$$

where S_{n-1} denotes the unit sphere in $\mathbf{R}^{n-1}$ and $d\sigma_{n-1}$ denotes normalized surface-area measure on S_{n-1}. The Riemann-Lebesgue Lemma then shows that $\int_S f_m g\, d\sigma \to 0$ as $m \to \infty$. In particular, taking $g = P(x, \cdot)$ for $x \in B$, we see that $P[f_m] \to 0$ pointwise on B.

Because $|f_m| \equiv 1$ on S, we have $|P[f_m]| \le 1$ on B for each m. Thus by Theorem 2.6, every subsequence of $(P[f_m])$ contains a subsequence converging uniformly on compact subsets of B. Because we already know that $P[f_m] \to 0$ pointwise on B, we must have $P[f_m] \to 0$ uniformly on compact subsets of B. ■

The harmonic functions of Lemma 8.18 extend continuously to $\overline{B}$ with boundary values of modulus one everywhere on S, yet converge uniformly to zero on compact subsets of B. In this they resemble the harmonic functions z^m in the unit disk of the complex plane.

8.19 Theorem: *Let $\alpha\colon [0,1) \to [1,\infty)$ be an increasing function with $\alpha(r) \to \infty$ as $r \to 1$. Then there exists a harmonic function u on B such that*

(a) $|u(r\zeta)| < \alpha(r)$ *for all $r \in [0,1)$ and all $\zeta \in S$;*

(b) *at every point of S, u fails to have a finite radial limit.*

PROOF: Choose an increasing sequence of numbers $s_m \in [0,1)$ such that $\alpha(s_m) > m+1$. From the sequence $(P[f_m])$ of Lemma 8.18, choose a subsequence (v_m) with $|v_m| < 2^{-m}$ on $s_m B$. Suppose $r \in [s_m, s_{m+1})$. Because each v_m is bounded by 1 on B, we have

$$\begin{aligned}\sum_{k=1}^{\infty} |v_k(r\zeta)| &= \sum_{k=1}^{m} |v_k(r\zeta)| + \sum_{k=m+1}^{\infty} |v_k(r\zeta)| \\ &< m + 2^{-(m+1)} + 2^{-(m+2)} + \cdots \\ &< m+1 \\ &< \alpha(s_m) \\ &\le \alpha(r).\end{aligned}$$

Thus $\sum |v_m(r\zeta)| < \alpha(r)$ for all $r \in (0,1)$ and all $\zeta \in S$; furthermore, $\sum |v_m|$ converges uniformly on compact subsets of B.

From the sequence (v_m) we inductively extract a subsequence (u_m) in the following manner. Set $u_1 = v_1$. Because u_1 is continuous on $\overline{B}$,

we may choose $r_1 \in [0,1)$ such that $|u_1(r\zeta) - u_1(\zeta)| < 1/4$ for all $r \in [r_1, 1]$ and all $\zeta \in S$. Suppose we have chosen $u_1, u_2, \dots, u_m$ from $v_1, v_2, \dots$ and that we have radii $0 < r_1 < \cdots < r_m < 1$ such that

8.20 $$\sum_{k=1}^{m} |u_k(r\zeta) - u_k(s\zeta)| < 1/4 \quad \text{for all } r, s \in [r_m, 1],\ \zeta \in S.$$

We then select u_{m+1} such that $|u_{m+1}| < 2^{-(m+1)}$ on $r_m\overline{B}$. Now choose $r_{m+1} \in (r_m, 1)$ so that 8.20 holds with $m+1$ in place of m. The radius r_{m+1} can be chosen since each u_k is continuous on $\overline{B}$.

Having obtained the subsequence (u_m) from (v_m) (as well as the accompanying sequence (r_m) of radii), we define

$$u = \sum_{m=1}^{\infty} u_m.$$

From the first paragraph of the proof we know that $|u(r\zeta)| < \alpha(r)$ for all $r \in [0,1)$, and that $\sum u_m$ converges uniformly on compact subsets of B, which implies that u is harmonic on B.

We now show that at each point of S, u fails to have a radial limit. (Here is where we use the fact that $|u_m| \equiv 1$ on S for every m.) We have

$$\begin{aligned}
|u(r_{m+1}\zeta) - u(r_m\zeta)| &\geq |u_{m+1}(r_{m+1}\zeta) - u_{m+1}(r_m\zeta)| \\
&\qquad - \sum_{k \neq m+1} |u_k(r_{m+1}\zeta) - u_k(r_m\zeta)| \\
&\geq |u_{m+1}(\zeta)| - |u_{m+1}(r_{m+1}\zeta) - u_{m+1}(\zeta)| \\
&\qquad - |u_{m+1}(r_m\zeta)| - 1/4 - 2\sum_{m+2}^{\infty} 2^{-k} \\
&\geq 1 - 1/4 - 2^{-(m+1)} - 1/4 - 2\sum_{m+2}^{\infty} 2^{-k} \\
&\geq 1/2 - 2\sum_{m+1}^{\infty} 2^{-k}.
\end{aligned}$$

Thus for each $\zeta \in S$, the sequence $(u(r_m\zeta))$ fails to have a finite limit as $m \to \infty$, which implies that u fails to have a finite radial limit at ζ. ■

8.21 Corollary: *There is a function u belonging to $\bigcap_{p<\infty} b^p(B)$ such that at every point of S, u fails to have a radial limit.*

PROOF: Let $\alpha(r) = 1 + \log \frac{1}{1-r}$, and let u be the corresponding function guaranteed by Theorem 8.19. Integrating in polar coordinates (1.5), we easily check that u belongs to $b^p(B)$ for every $p \in [1, \infty)$. ∎

The Reproducing Kernel of the Upper Half-Space

The goal of this section is to find an explicit formula for the reproducing kernel of the upper half-space. A well-motivated, although computationally tedious, method of deriving this formula is given in Exercise 24 of this chapter. We will present a slicker method relying on the magic of integration by parts.

As we did for B, we will derive the reproducing kernel of H in terms of the Poisson kernel. Recall that for $z \in H$ and $t \in \mathbf{R}^{n-1}$, the Poisson kernel for H is the function

$$P_H(z,t) = \frac{2}{nV(B)} \frac{z_n}{|z-t|^n}.$$

For $w \in \mathbf{R}^n$, define $\overline{w} = (w_1, \dots, w_{n-1}, -w_n)$; note that $\overline{w}$ is the usual complex conjugate of w on $\mathbf{R}^2 = \mathbf{C}$. We now extend the domain of P_H by defining

8.22
$$P_H(z,w) = \frac{2}{nV(B)} \frac{z_n + w_n}{|z-\overline{w}|^n}$$

for $z \neq \overline{w}$. Note that $P_H(z,w) = P_H(w,z)$ and $P_H(z+(0,r), w) = P_H(z, w+(0,r))$ for $r \in \mathbf{R}$ whenever these expressions make sense. Thus

$$\begin{aligned} P_H(z,w) &= P_H(w,z) \\ &= P_H(w+(0,z_n), z-(0,z_n)) \end{aligned}$$

for all $z, w \in H$. Thus $P_H(z,\cdot)$ is harmonic on $\{w \in \mathbf{R}^n : w_n > -z_n\}$ for each $z \in H$ (being the translate of a harmonic function).

Before proving the main result of this section, Theorem 8.24, we prove an analogue of Lemma 8.8 for H.

8.23 Lemma: *The set of functions that are harmonic and square integrable on a half-space larger than H is dense in $b^2(H)$.*

PROOF: Let $u \in b^2(H)$. For $\delta > 0$, the function $z \mapsto u(z + (0, \delta))$ belongs to $b^2(\{z \in \mathbf{R}^n : z_n > -\delta\})$. But the functions $u(z + (0, \delta))$ converge to $u(z)$ in $L^2(H, dV)$ as $\delta \to 0$. (This follows by uniform continuity if u is continuous and has compact support in H; the set of such functions is dense in $L^2(H, dV)$.) ∎

Now we can give an explicit formula for the reproducing kernel of the upper half-space.

8.24 Theorem: *For all $z, w \in H$,*

$$R_H(z, w) = -2\frac{\partial}{\partial w_n} P_H(z, w) = \frac{4}{nV(B)} \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}}.$$

PROOF: The second equality follows, with some simple calculus, from 8.22. Note that this equality implies $\partial P_H(z, w)/\partial w_n$ belongs to $b^2(H)$ for each fixed $z \in H$ (see Exercise 1 in Appendix A). The remainder of the proof will be devoted to showing that the first equality holds.

Fix $z \in H$. Suppose $\delta > 0$ and $u \in b^2(\{w \in \mathbf{R}^n : w_n > -\delta\})$. Then

8.25
$$\begin{aligned}\int_H u(w) \frac{\partial}{\partial w_n} P_H(z, w)\, dV(w) \\ = \int_{\mathbf{R}^{n-1}} \int_0^\infty u(x, y) \frac{\partial}{\partial y} P_H(z, (x, y))\, dy\, dx.\end{aligned}$$

Now, u is bounded and harmonic on $\overline{H}$ by 8.1. Thus, after integrating by parts in the inner integral, the right side of 8.25 becomes

$$\begin{aligned}&- \int_{\mathbf{R}^{n-1}} u(x, 0) P_H(z, x)\, dx \\ &- \int_{\mathbf{R}^{n-1}} \int_0^\infty \left[\frac{\partial}{\partial y} u(x, y)\right] P_H(z, (x, y))\, dx\, dy,\end{aligned}$$

which equals

8.26 $$-u(z) - \int_0^\infty \int_{\mathbf{R}^{n-1}} \left[\frac{\partial}{\partial y} u(x,y)\right] P_H(z,(x,y))\,dx\,dy.$$

Notice that we reversed the order of integration to arrive at 8.26. This is permissible if the integrand in 8.26 is integrable over H. To verify this, note that by Corollary 8.2 there exists a constant $C < \infty$ such that

$$\left|\frac{\partial}{\partial y} u(x,y)\right| \le \frac{C}{(y+\delta)^{1+n/2}}.$$

Note also that

8.27 $$P_H(z,(x,y)) = P_H(z+(0,y),(x,0)),$$

which implies $\int_{\mathbf{R}^{n-1}} P_H(z,(x,y))\,dx = 1$ for each $y > 0$. The reader can now easily verify that the integrand in 8.26 is integrable over H.

For each $y > 0$, the term in brackets in 8.26 is the restriction to $\mathbf{R}^{n-1}$ of the function $w \mapsto D_n u(w + (0,y))$, which is bounded and harmonic on $\overline{H}$. Thus by 8.27, the integral over $\mathbf{R}^{n-1}$ in 8.26 equals $(D_n u)(z + (0,2y))$. Therefore 8.26 equals

$$-u(z) - \int_0^\infty (D_n u)(z+(0,2y))\,dy = -u(z)/2,$$

where the last equality holds because $u(z + (0,2y)) \to 0$ as $y \to \infty$ (by 8.1).

Let $F(w) = -2\partial P_H(z,w)/\partial w_n$. We have shown that $u(z) = \langle u, F\rangle$ whenever u is harmonic and square integrable on a half-space larger than H. The set of such functions u is dense in $b^2(H)$ (Lemma 8.23), and thus the proof is complete. ■

Exercises

1. Prove that $b^p(\mathbf{R}^n) = \{0\}$.

2. Suppose that $u \in b^p(\Omega)$. Prove that $d(x, \partial\Omega)^{n/p}|u(x)| \to 0$ as $x \to \partial\Omega$.

3. Prove that if $u \in b^p(\{x \in \mathbf{R}^n : |x| > 1\})$, then u is harmonic at ∞.

4. Suppose $u \in b^p(H)$ and $y > 0$. Prove that $u(x, y) \to 0$ as $|x| \to \infty$ in $\mathbf{R}^{n-1}$.

5. Prove that if u is a harmonic function on $\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} |u(x)|(1 + |x|)^{\lambda} \, dV(x) < \infty$$

for some $\lambda \in \mathbf{R}$, then u is a polynomial.

6. (a) Assume that $n > 2$ and $p \geq n/(n-2)$. Prove that if u is in $b^p(B \setminus \{0\})$, then u has a removable singularity at 0.

   (b) Show that the constant $n/(n-2)$ in part (a) is sharp.

   (c) Show that there exists a function in $\bigcap_{p<\infty} b^p(B_2 \setminus \{0\})$ that fails to have a removable singularity at 0.

7. Prove that $b^p(\mathbf{R}^n \setminus \{0\}) = \{0\}$.

8. Prove that

$$R_{r\Omega+a}(x, y) = r^{-n} R_{\Omega}\Big(\frac{x-a}{r}, \frac{y-a}{r}\Big)$$

for all $r > 0$, $a \in \mathbf{R}^n$.

9. Prove that $\|R_{\Omega}(x, \cdot)\|_{b^2} \leq \dfrac{1}{\sqrt{V(B) d(x, \partial\Omega)^n}}$.

10. Suppose $\Omega_1 \subset \Omega_2 \subset \cdots$ is an increasing sequence of open subsets of $\mathbf{R}^n$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Prove that

$$R_{\Omega}(x, y) = \lim_{k \to \infty} R_{\Omega_k}(x, y)$$

for all $x, y \in \Omega$.

11. Suppose $a_1, \ldots, a_m$ are points in Ω. Let A be the m-by-m matrix whose entry in row j, column k, equals $R_\Omega(a_j, a_k)$. Prove that A is positive semidefinite.

12. Show that the harmonic Bloch space is properly contained in $b^p(B)$ for every $p < \infty$. (See Exercise 11, Chapter 2, for the definition of the harmonic Bloch space.)

13. Show that
$$u(x) = \int_B u(y) R_B(x, y)\, dV(y)$$
for all $u \in b^p(B)$ and for all $p \in [1, \infty)$.

14. Assume $n = 2$, and set $u_k(re^{i\theta}) = r^{|k|}e^{ik\theta}$, $k = 0, \pm 1, \ldots$. Find constants c_k so that $\{c_k u_k\}$ is an orthonormal basis of $b^2(B)$, and then use Proposition 8.4(b) to find a formula for the reproducing of kernel of B_2.

15. (a) Prove there are positive constants C_1, C_2 such that
$$\frac{C_1}{(1 - |x|)^{n/2}} \le \|R_B(x, \cdot)\|_{b^2} \le \frac{C_2}{(1 - |x|)^{n/2}}$$
for all $x \in B$.

    (b) Find an estimate analogous to (a) for $\|R_H(z, \cdot)\|_{b^2}$.

16. Show that $R_B(x, \cdot)/\|R_B(x, \cdot)\|_{b^2}$ converges to 0 weak* in $b^2(B)$ as $|x| \to 1$.

17. Show that
$$Q_B[{x_1}^2] = \frac{1}{n+2} + {x_1}^2 - \frac{\|x\|^2}{n}.$$

18. Prove that if $p \in \mathcal{P}_m(\mathbf{R}^n)$ and $Q_B[p] = 0$, then $p = 0$.

19. Prove that if $p \in \mathcal{P}_m(\mathbf{R}^n)$ and $P[p] = Q_B[p]$, then p is harmonic.

20. Fix $\zeta \in S$. Show that $P(\cdot, \zeta) \in b^p(B)$ for $p < n/(n-1)$. Also show that $P(\cdot, \zeta) \notin b^{n/(n-1)}(B)$.

21. Show that $h^p(B) \subset b^q(B)$ for $q < pn/(n-1)$.

22. Prove that every infinite-dimensional closed subspace of $b^2(B)$ contains a function not in $h^2(B)$.

23. Show that functions in $b^p(B)$ belong to appropriate Hardy spaces of balls internally tangent to B. More precisely, suppose that $1 \le q < (n-1)p/(2n)$ and $u \in b^p(B)$. Prove that if $a \in B \setminus \{0\}$, then the function $x \mapsto u(a + (1 - |a|)x)$ is in $h^q(B)$.

24. Derive the formula for the reproducing kernel R_H (Theorem 8.24) by writing $H = \bigcup_{k=1}^{\infty} B(k\mathbf{N}, k)$ and then using Exercise 10 of this chapter and 8.13.

25. Show that every positive harmonic function on B is in $b^1(B)$. Are there any positive harmonic functions on H that are in $b^1(H)$?

26. Suppose $\delta > 0$ and $u \in b^p(\{z \in \mathbf{R}^n : z_n > -\delta\})$. Show that $D^\alpha u \in b^p(H)$ for every multi-index α.

CHAPTER 9

The Decomposition Theorem

If $K \subset \Omega$ is compact and u is harmonic on $\Omega \setminus K$, then u might be badly behaved near both ∂K and $\partial\Omega$; see, for example, Theorem 11.18. In this chapter we will see that u is the sum of two harmonic functions, one extending harmonically across ∂K, the other extending harmonically across $\partial\Omega$. More precisely, u has a decomposition of the form

$$u = v + w$$

on $\Omega \setminus K$, where v is harmonic on Ω and w is harmonic on $\mathbf{R}^n \setminus K$. Furthermore, there is a canonical choice for w that makes this decomposition unique.

This result, which we call the decomposition theorem, has many applications. In this chapter we will use it to prove a generalization of Bôcher's Theorem, to show that bounded harmonic functions extend harmonically across smooth sets of dimension $n - 2$, and to prove the logarithmic conjugation theorem. In Chapter 10, we will use the decomposition theorem to obtain a "Laurent" series expansion for harmonic functions on annular domains in $\mathbf{R}^n$.

The Fundamental Solution of the Laplacian

We have already seen how important the functions $|x|^{2-n}$ $(n > 2)$ and $\log|x|$ $(n = 2)$ are to harmonic function theory. Another illustra-

tion of their importance is that they give rise to integral operators that invert the Laplacian; we will need these operators in the proof of the decomposition theorem.

The *support* of a function g on $\mathbf{R}^n$, denoted $\operatorname{supp} g$, is the closure of the set $\{x \in \mathbf{R}^n : g(x) \neq 0\}$. We let $C_c^k = C_c^k(\mathbf{R}^n)$ denote the set of functions in $C^k(\mathbf{R}^n)$ that have compact support. We will frequently use the abbreviation dy for the usual volume measure $dV(y)$.

We now show how g can be reconstructed from Δg if $g \in C_c^2$.

9.1 Theorem ($n > 2$): *If $g \in C_c^2$, then*

$$g(x) = \frac{1}{(2-n)nV(B)} \int_{\mathbf{R}^n} (\Delta g)(y)|x-y|^{2-n}\,dy$$

for every $x \in \mathbf{R}^n$.

9.2 Theorem ($n = 2$): *If $g \in C_c^2$, then*

$$g(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} (\Delta g)(y) \log|x-y|\,dy$$

for every $x \in \mathbf{R}^2$.

PROOF: We present the proof for $n > 2$, leaving the minor modifications needed for $n = 2$ to the reader (Exercise 1 of this chapter). Note first that the function $|x|^{2-n}$ is locally integrable on $\mathbf{R}^n$ (use polar coordinates 1.5).

Fix $x \in \mathbf{R}^n$. Choose r large enough so that $B(0,r)$ contains both x and $\operatorname{supp} g$. For small $\varepsilon > 0$, set $\Omega_\varepsilon = B(0,r) \setminus \overline{B}(x,\varepsilon)$. Because g is supported in $B(0,r)$,

$$\int_{\mathbf{R}^n} (\Delta g)(y)|x-y|^{2-n}\,dy = \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (\Delta g)(y)|x-y|^{2-n}\,dy.$$

Now apply Green's identity (1.1)

$$\int_\Omega (u\Delta g - g\Delta u)\,dV = \int_{\partial\Omega} (uD_{\mathbf{n}}g - gD_{\mathbf{n}}u)\,ds,$$

with $u(y) = |x-y|^{2-n}$ and $\Omega = \Omega_\varepsilon$. Since $g = 0$ near $\partial B(0,r)$, only the surface integral over $\partial B(x,\varepsilon)$ comes into play. Recalling that the

unnormalized surface area of S is $nV(B)$ (see A.2 in Appendix A), we calculate that

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} (\Delta g)(y)|x - y|^{2-n}\,dy = (2-n)nV(B)g(x),$$

which gives the desired conclusion. ∎

For $x \in \mathbf{R}^n \setminus \{0\}$, set

$$F(x) = \begin{cases} [(2-n)nV(B)]^{-1}|x|^{2-n} & \text{if } n > 2 \\ (2\pi)^{-1}\log|x| & \text{if } n = 2. \end{cases}$$

The function F is called the *fundamental solution* of the Laplacian; it serves as the kernel of an integral operator that inverts the Laplacian on C_c^2. To see this, define

9.3 $$(Tg)(x) = \int_{\mathbf{R}^n} g(y)F(x-y)\,dy = \int_{\mathbf{R}^n} g(x-y)F(y)\,dy$$

for $g \in C_c$. Now suppose $g \in C_c^2$. Then $T(\Delta g) = g$ by 9.1 or 9.2. On the other hand, differentiation under the integral sign on the right side of 9.3 shows that $\Delta(Tg) = T(\Delta g)$; applying 9.1 or 9.2 again, we see that $\Delta(Tg) = g$. Thus $T \circ \Delta = \Delta \circ T = I$, the identity operator, on the space C_c^2; in other words, $T = \Delta^{-1}$ on C_c^2.

We can now solve the inhomogeneous equation

9.4 $$\Delta u = g$$

for any $g \in C_c^2$; we simply take $u = Tg$. Equation 9.4 is often referred to as *Poisson's equation*.

Decomposition of Harmonic Functions

The reader is already familiar with a result from complex analysis that can be interpreted as a decomposition theorem. Specifically, suppose $0 < r < R < \infty$, $K = \overline{B}(0,r)$, and $\Omega = B(0,R)$. Assume f is holomorphic on the annulus $\Omega \setminus K$, and let $\sum_{-\infty}^{\infty} a_k z^k$ be the Laurent expansion of f on $\Omega \setminus K$. Setting $g(z) = \sum_0^\infty a_k z^k$ and $h(z) = \sum_{-\infty}^{-1} a_k z^k$, we see that $f = g + h$ on $\Omega \setminus K$, that g extends to be holomorphic on Ω,

and that h extends to be holomorphic on $(\mathbf{C} \cup \{\infty\}) \setminus K$. The Laurent series expansion therefore gives us a decomposition for holomorphic functions (in the special case of annular regions in $\mathbf{C}$). The decomposition theorem (9.6 and 9.7) is the analogous result for harmonic functions.

We will need a large supply of smooth functions in the proof of the decomposition theorem; the following lemma provides what we want.

9.5 Lemma: *Suppose $K \subset \Omega$ is compact. Then there exists a function $\varphi \in C_c^\infty(\mathbf{R}^n)$ such that $\varphi \equiv 1$ on K, $\operatorname{supp} \varphi \subset \Omega$, and $0 \le \varphi \le 1$ on $\mathbf{R}^n$.*

PROOF: Define a C^∞-function f on $\mathbf{R}$ by setting

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \le 0, \end{cases}$$

and define a function $\psi \in C_c^\infty(\mathbf{R}^n)$ by setting $\psi(y) = cf(1 - 2|y|^2)$, where the constant c is chosen so that $\int_{\mathbf{R}^n} \psi(y)\,dy = 1$. Note that $\operatorname{supp} \psi \subset B$.

For $r > 0$, let $\psi_r(y) = r^{-n}\psi(y/r)$. Observe that $\operatorname{supp} \psi_r \subset rB$ and that $\int_{\mathbf{R}^n} \psi_r(y)\,dy = 1$. Now set $r = d(K, \partial\Omega)/3$ and define

$$\omega = \{x \in \Omega : d(x, K) < r\}.$$

Finally, put

$$\varphi(x) = \int_\omega \psi_r(x - y)\,dy$$

for $x \in \mathbf{R}^n$. Differentiation under the integral sign above shows that $\varphi \in C^\infty$. Clearly $0 \le \varphi \le 1$ on $\mathbf{R}^n$. Because $\psi_r(x - y)$ is supported in $B(x, r)$, we have $\varphi(x) = 1$ whenever $x \in K$ and $\varphi(x) = 0$ whenever $d(x, K) > 2r$. ■

We now prove the decomposition theorem; the $n > 2$ case differs from the $n = 2$ case, so we state the two results separately.

9.6 Decomposition Theorem ($n > 2$): *Let K be a compact subset of Ω. If u is harmonic on $\Omega \setminus K$, then u has a unique decomposition of the form*

$$u = v + w,$$

where v is harmonic on Ω and w is a harmonic function on $\mathbf{R}^n \setminus K$ satisfying $\lim_{x\to\infty} w(x) = 0$.

9.7 Decomposition Theorem ($n = 2$): *Let K be a compact subset of Ω. If u is harmonic on $\Omega \setminus K$, then u has a unique decomposition of the form*

$$u = v + w,$$

where v is harmonic on Ω and w is a harmonic function on $\mathbf{R}^2 \setminus K$ satisfying $\lim_{x\to\infty} w(x) - b\log|x| = 0$ *for some constant b.*

PROOF: We present the proof for $n > 2$ (Theorem 9.6), leaving the changes needed for $n = 2$ to the reader (Exercise 3 of this chapter).

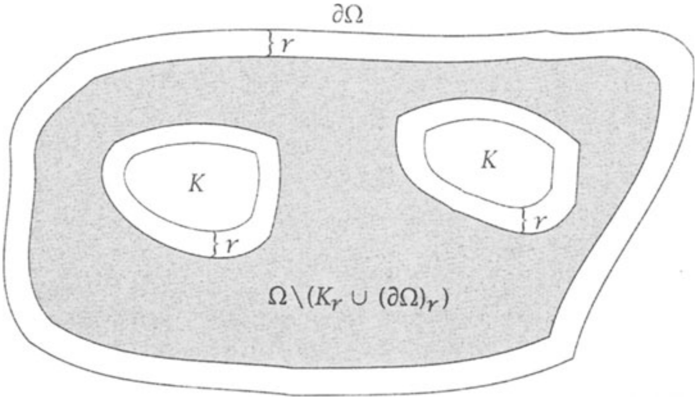
As a notation convenience, for E any subset of $\mathbf{R}^n$ and $r > 0$, let $E_r = \{x \in \mathbf{R}^n : d(x, E) < r\}$.

Suppose first that Ω is a bounded open subset of $\mathbf{R}^n$. Choose r small enough so that K_r and $(\partial\Omega)_r$ are disjoint. By Lemma 9.5, there is a function $\varphi_r \in C_c^\infty(\mathbf{R}^n)$ supported in $\Omega \setminus K$ such that $\varphi_r \equiv 1$ on $\Omega \setminus (K_r \cup (\partial\Omega)_r)$; Figure 9.8 may be helpful.

For $x \in \Omega \setminus (K_r \cup (\partial\Omega)_r)$, apply Theorem 9.1 to the function $u\varphi_r$, which can be thought of as a function in $C_c^\infty(\mathbf{R}^n)$, to obtain

$$\begin{aligned} u(x) &= (u\varphi_r)(x) \\ &= \frac{1}{(2-n)nV(B)} \int_{\mathbf{R}^n} \Delta(u\varphi_r)(y)|x-y|^{2-n}\,dy \\ &= \frac{1}{(2-n)nV(B)} \int_{(\partial\Omega)_r} \Delta(u\varphi_r)(y)|x-y|^{2-n}\,dy \\ &\quad + \frac{1}{(2-n)nV(B)} \int_{K_r} \Delta(u\varphi_r)(y)|x-y|^{2-n}\,dy \\ &= v_r(x) + w_r(x), \end{aligned}$$

where $v_r(x)$ is $[(2-n)nV(B)]^{-1}$ times the integral over $(\partial\Omega)_r$ and $w_r(x)$ is $[(2-n)nV(B)]^{-1}$ times the integral over K_r. Differentiation



9.8 *$\varphi_r \equiv 1$ on the shaded region.*

under the integral sign shows that v_r is harmonic on $\Omega \setminus (\partial\Omega)_r$ and that w_r is harmonic on $\mathbf{R}^n \setminus K_r$. We also see that $w_r(x) \to 0$ as $x \to \infty$.

Suppose now that $s < r$. Then, as in the previous paragraph, we obtain the decomposition $u = v_s + w_s$ on $\Omega \setminus (K_s \cup \partial\Omega_s)$. We claim that $v_r = v_s$ on $\Omega \setminus (\partial\Omega)_r$ and $w_r = w_s$ on $(\mathbf{R}^n \cup \{\infty\}) \setminus K_r$. To see this, note that if $x \in \Omega \setminus (K_r \cup (\partial\Omega)_r)$, then $v_r(x) + w_r(x) = v_s(x) + w_s(x)$ (because both sides equal $u(x)$). Thus, $w_r - w_s$ is a harmonic function on $\mathbf{R}^n \setminus K_r$ that extends to be harmonic on $\mathbf{R}^n$ ($w_r - w_s$ agrees with $v_s - v_r$ near K_r). Because both w_r and w_s tend to 0 at infinity, Liouville's Theorem (2.1) implies that $w_r - w_s \equiv 0$. Thus $w_r = w_s$ and $v_r = v_s$ on $\Omega \setminus (K_r \cup (\partial\Omega)_r)$, as claimed.

For $x \in \Omega$, we may thus set $v(x) = v_r(x)$ for all r small enough so that $x \in \Omega \setminus (\partial\Omega)_r$. Similarly, for $x \in \mathbf{R}^n \setminus K$, we set $w(x) = w_r(x)$ for small r. We have arrived at the desired decomposition $u = v + w$.

Now suppose that Ω is unbounded and u is harmonic on $\Omega \setminus K$. Choose R large enough so that $K \subset B(0, R)$ and let $\omega = \Omega \cap B(0, R)$. Observe that K is a compact subset of the bounded open set ω and that u is harmonic on $\omega \setminus K$. Applying the result just proved for bounded open sets, we have

$$u(x) = \tilde{v}(x) + w(x)$$

for $x \in \omega \setminus K$, where $\tilde{v}$ is harmonic on ω and w is a harmonic function on $\mathbf{R}^n \setminus K$ satisfying $\lim_{x\to\infty} w(x) = 0$. Notice that the difference $u - w$ is harmonic on $\Omega \setminus K$ and extends harmonically across K because it

agrees with $\tilde{v}$ near K. Set $v = u - w$; the sum $v + w$ is then the desired decomposition of u.

Finally, the proof of the uniqueness of the decomposition is similar to the proof given above that $w_r = w_s$ and $v_r = v_s$ on $\Omega \setminus (K_r \cup (\partial\Omega)_r)$. ■

Note that the function w of Theorem 9.6 is harmonic at ∞, by Theorem 4.8. Note also that if u is real valued, then the functions v and w appearing in the decompositon of u (Theorem 9.6 or Theorem 9.7) will also be real valued. This can be proved either by looking at the proofs of Theorem 9.6 and Theorem 9.7 or by taking the real parts of both sides of the decomposition $u = v + w$ and using the uniqueness of the decomposition.

Bôcher's Theorem Revisited

The remainder of this chapter consists of applications of the decomposition theorem. We begin by using it to obtain Bôcher's Theorem (3.9) as a consequence of Liouville's Theorem in the $n > 2$ setting.

9.9 Bôcher's Theorem ($n > 2$): *Let $a \in \Omega$. If u is harmonic on $\Omega \setminus \{a\}$ and positive near a, then there is a harmonic function v on Ω and a constant $b \geq 0$ such that*

$$u(x) = v(x) + b|x - a|^{2-n}$$

for all $x \in \Omega \setminus \{a\}$.

PROOF: Without loss of generality we can assume that u is real valued and $a = 0$. By the decomposition theorem (9.6), we can write $u = v + w$, where v is harmonic on Ω, w is harmonic on $\mathbf{R}^n \setminus \{0\}$, and $\lim_{x\to\infty} w(x) = 0$. We will complete the proof by showing that $w(x) = b|x|^{2-n}$ for some constant $b \geq 0$.

Because u is positive near 0 and v is bounded near 0, $w = u - v$ is bounded below near 0. Let $\varepsilon > 0$ and set $h(x) = w(x) + \varepsilon|x|^{2-n}$. Then $\lim_{x\to 0} h(x) = \infty$ and $\lim_{x\to\infty} h(x) = 0$, so the minimum principle (1.10) implies that $h \geq 0$ on $\mathbf{R}^n \setminus \{0\}$. Letting $\varepsilon \to 0$, we conclude that $w \geq 0$ on $\mathbf{R}^n \setminus \{0\}$.

Because w tends to zero at ∞, the Kelvin transform $K[w]$ has a removable singularity at 0 (see Exercise 2(a) in Chapter 2). Thus $K[w]$

extends to be nonnegative and harmonic on all of $\mathbf{R}^n$. By Liouville's Theorem for positive harmonic functions (3.1), $K[w] = b$ for some constant $b \geq 0$. Therefore $w(x) = b|x|^{2-n}$, completing the proof. ■

The preceding argument does not yield a proof of Bôcher's Theorem when $n = 2$. (One difficulty is that the function w provided by the decomposition theorem no longer vanishes at ∞.) We can, however, still use the decomposition theorem to prove Bôcher's Theorem in the $n = 2$ case. We will actually obtain a generalized version of Bôcher's Theorem. Our proof relies on the following improvement of Liouville's Theorem for positive harmonic functions (3.1).

9.10 Generalized Liouville Theorem: *Suppose that u is a real-valued harmonic function on $\mathbf{R}^n$ and*

$$\liminf_{x \to \infty} \frac{u(x)}{|x|} \geq 0.$$

Then u is constant on $\mathbf{R}^n$.

PROOF: Fix $x \in \mathbf{R}^n$, let $\varepsilon > 0$ be arbitrary, and choose $r > |x|$ such that $u(y)/|y| \geq -\varepsilon$ whenever $|y| > r - |x|$. By the volume version of the mean-value property,

$$u(x) - u(0) = \frac{1}{V(B(0,r))}\Big[\int_{B(x,r)} u\, dV - \int_{B(0,r)} u\, dV\Big].$$

Let $\mathcal{D}_r$ denote the symmetric difference of the balls $B(x,r)$ and $B(0,r)$ (see Figure 3.2) and let A_r denote the annulus $B(0,r+|x|)\setminus B(0,r-|x|)$. Then

$$\begin{aligned} |u(x) - u(0)| &\leq \frac{1}{V(B(0,r))}\int_{\mathcal{D}_r} |u|\, dV \\ &\leq \frac{1}{V(B(0,r))}\int_{A_r} |u|\, dV. \end{aligned}$$

For every y in the annulus A_r over which the last integral is taken, we have

$$|u(y)| \leq 2\varepsilon|y| + u(y) \leq 4\varepsilon r + u(y),$$

where the first inequality is trivial when $u(y) \geq 0$ and follows from our choice of r when $u(y) < 0$. Combining the last two sets of inequalities, we have

$$|u(x) - u(0)| \le \frac{1}{V(B(0,r))}\left[4\varepsilon r V(A_r) + \int_{A_r} u\, dV\right]$$
$$= (4\varepsilon r + u(0))\frac{(r+|x|)^n - (r-|x|)^n}{r^n}.$$

Take the limit as $r \to \infty$, getting

$$|u(x) - u(0)| \le 8\varepsilon n|x|.$$

Now take the limit as $\varepsilon \to 0$, getting $u(x) = u(0)$, as desired. ■

Note that u satisfies the hypothesis of the Generalized Liouville Theorem if and only if

$$\liminf_{x \to 0} |x|^{n-1} K[u](x) \ge 0,$$

which explains the hypothesis of the following result.

9.11 Generalized Bôcher Theorem: *Let $a \in \Omega$. Suppose that u is a real-valued harmonic function on $\Omega \setminus \{a\}$ and*

$$\liminf_{x \to a} |x - a|^{n-1} u(x) \ge 0.$$

Then there is a harmonic function v on Ω and a constant $b \in \mathbf{R}$ such that

$$u(x) = \begin{cases} v(x) + b\log|x-a| & \text{if } n = 2 \\ v(x) + b|x-a|^{2-n} & \text{if } n > 2 \end{cases}$$

for all $x \in \Omega \setminus \{a\}$.

PROOF: We will assume that $n = 2$, leaving the easier $n > 2$ case as an exercise for the reader.

Without loss of generality, we may assume that $a = 0$. Because u is harmonic on $\Omega \setminus \{0\}$, it has a decomposition

$$u = v + w,$$

where v is harmonic on Ω and w is a harmonic function on $\mathbf{R}^2 \setminus \{0\}$ satisfying $\lim_{x \to 0}(w(x) - b\log|x|) = 0$ for some constant $b \in \mathbf{R}$. Because v is continuous at 0, our hypothesis on u implies that

$$\liminf_{x\to 0} |x|w(x) \geq 0.$$

Now set $h(x) = w(x) - b\log|x|$ for $x \in \mathbf{R}^2 \setminus \{0\}$, and observe that the Kelvin transform $\mathcal{K}[h]$ has a removable singularity at 0. Moreover,

$$\begin{aligned}\liminf_{x\to\infty} \frac{\mathcal{K}[h](x)}{|x|} &= \liminf_{x\to\infty} \frac{w(x/|x|^2) - b\log|1/x|}{|x|}\\ &= \liminf_{x\to 0} |x|w(x)\\ &\geq 0.\end{aligned}$$

Thus, by the the Generalized Liouville Theorem (9.10), $\mathcal{K}[h]$ must be constant; in fact, it must be zero because its value at 0 is 0. Hence $h = 0$, which implies that $w(x) = b\log|x|$. Thus u has the desired form. ∎

The preceding proof shows how the Generalized Bôcher Theorem follows from the Generalized Liouville Theorem. It is even easier to show that the Generalized Liouville Theorem follows from the Generalized Bôcher Theorem (Exercise 8 of this chapter); hence, these results are equivalent. The authors first learned of these generalizations of Bôcher's and Liouville's Theorems in [1] and [10].

Removable Sets for Bounded Harmonic Functions

Let $h^\infty(\Omega)$ denote the collection of bounded harmonic functions on Ω. We say that a compact set $K \subset \Omega$ is *h^∞-removable* for Ω if every bounded harmonic function on $\Omega \setminus K$ extends to be harmonic on Ω. The following theorem shows that if K is h^∞-removable for some Ω containing K, then K is h^∞-removable for every Ω containing K. Note that by Liouville's Theorem, K is h^∞-removable for $\mathbf{R}^n$ if and only if every bounded harmonic function on $\mathbf{R}^n \setminus K$ is constant.

9.12 Theorem: *Let K be a compact subset of Ω. Then K is h^∞-removable for Ω if and only if K is h^∞-removable for $\mathbf{R}^n$.*

PROOF: If K is h^∞-removable for Ω, then clearly K is h^∞-removable for $\mathbf{R}^n$.

To prove the converse, we use the decomposition theorem. The $n > 2$ case is easy. Suppose that K is h^∞-removable for $\mathbf{R}^n$ and that u is bounded and harmonic on $\Omega \setminus K$. Let $u = v + w$ be the decomposition given by (9.6). Because $n > 2$, $w(x) \to 0$ as $x \to \infty$. The boundedness near K of $w = u - v$ thus shows that w is bounded and harmonic on $\mathbf{R}^n \setminus K$. By hypothesis, w extends to be harmonic on $\mathbf{R}^n$, and thus $w \equiv 0$ by Liouville's Theorem. Hence $u = v$, and thus u extends to be harmonic on Ω, as desired.

The $n = 2$ case is more difficult (a rare occurrence); this is because w need not have limit 0 at ∞. We will show that if there is a bounded harmonic function on $\Omega \setminus K$ that does not extend to be harmonic on Ω, then there is a nonconstant bounded harmonic function on $\mathbf{R}^2 \setminus K$.

We may assume that each connected component of K is a point, in other words, that K is totally disconnected. Otherwise some component of K consists of more than one point. The Riemann Mapping Theorem then implies the existence of a holomorphic map of the Riemann sphere minus that component onto B_2, giving us a nonconstant bounded harmonic function on $\mathbf{R}^2 \setminus K$, as desired.

Let u be a bounded harmonic function on $\Omega \setminus K$ that does not extend to be harmonic on Ω. Then there exist distinct points x and y in K such that u does not extend harmonically to any neighborhood of x nor to any neighborhood of y. (If only one such point in K existed, then we would have found a nonremovable isolated singularity of a bounded harmonic function, contradicting Theorem 2.3.)

Having obtained x and y, observe that the total disconnectivity of K shows that there exist disjoint open sets Ω_x and Ω_y (open in $\mathbf{R}^2$), with $x \in \Omega_x$ and $y \in \Omega_y$, such that $K \subset \Omega_x \cup \Omega_y$.

Now u is harmonic on $(\Omega \cap \Omega_x) \setminus (\Omega_x \cap K)$, so by Theorem 9.7 we have the decomposition $u = v_x + w_x$, where v_x is harmonic on $\Omega \cap \Omega_x$ and w_x is harmonic on $\mathbf{R}^2 \setminus (\Omega_x \cap K)$, with $\lim_{z \to \infty} w_x(z) - b_x \log|z| = 0$ for some constant b_x. We also have a similar decomposition of u on $(\Omega \cap \Omega_y) \setminus (\Omega_y \cap K)$. Note that w_x is not constant, otherwise u would extend harmonically to a neighborhood of x. Note also that if b_x were 0, then w_x would be a nonconstant bounded harmonic function on $\mathbf{R}^2 \setminus K$, and we would be done; we may thus assume that b_x is nonzero.

Setting $h = w_y - (b_y / b_x) w_x$, we claim h is the desired nonconstant bounded harmonic function on $\mathbf{R}^2 \setminus K$. To see this, note that both w_x

and w_y are bounded near K, and $\lim_{z\to\infty} h(z) = 0$; this proves h is bounded and harmonic on $\mathbf{R}^2 \setminus K$. If h were constant, then w_y would extend harmonically to a neighborhood of y, which would mean that u would extend harmonically to a neighborhood of y, a contradiction. ∎

As an aside, note that the analogue of Theorem 9.12 for positive harmonic functions fails when $n = 2$: the compact set $\{0\}$ is removable for positive harmonic functions on $\mathbf{R}^2 \setminus \{0\}$ (see 3.3), but $\{0\}$ is not removable for positive harmonic functions on $B_2 \setminus \{0\}$.

Recall that if $K \subset \Omega$ is a single point, then K is h^∞-removable for Ω (Theorem 2.3). Our next theorem improves that result, stating (roughly) that if the dimension of K is less than or equal to $n - 2$, then K is h^∞-removable.

9.13 Theorem: *If $1 \le k \le n-2$ and $\Psi: \overline{B}_k \to \Omega$ is a C^1-map, then $\Psi(\overline{B}_k)$ is h^∞-removable for Ω.*

PROOF: By Theorem 9.12, we need only show that if u is bounded and harmonic on $\mathbf{R}^n \setminus \Psi(\overline{B}_k)$, then u is constant. Without loss of generality, we assume u is real valued. By Theorem 4.10, there is a constant L such that u has limit L at ∞. Let $\varepsilon > 0$ and set

$$v(x) = u(x) + \varepsilon \int_{B_k} |x - \Psi(y)|^{2-n}\, dV_k(y)$$

for $x \in \mathbf{R}^n \setminus \Psi(\overline{B}_k)$. Note that v is harmonic on $\mathbf{R}^n \setminus \Psi(\overline{B}_k)$ and that v has limit L at ∞. Suppose we know that

9.14 $$\int_{B_k} |x - \Psi(y)|^{2-n}\, dV_k(y) \to \infty \quad \text{as } x \to \Psi(\overline{B}_k).$$

The boundedness of u then shows $v(x) \to \infty$ as $x \to \Psi(\overline{B}_k)$. By the minimum principle, $v \ge L$ on $\mathbf{R}^n \setminus \Psi(\overline{B}_k)$. Letting $\varepsilon \to 0$, we conclude that $u \ge L$ on $\mathbf{R}^n \setminus \Psi(\overline{B}_k)$. A similar argument then gives $u \le L$ on $\mathbf{R}^n \setminus \Psi(\overline{B}_k)$, so that u is constant, as desired. In other words, to complete the proof, we need only show that 9.14 holds.

To prove 9.14, first suppose that $x \in \Psi(\overline{B}_k)$. Then $x = \Psi(z)$ for some $z \in \overline{B}_k$. Because Ψ has a continuous derivative, there is a constant $C \in (0, \infty)$ such that $|\Psi(z) - \Psi(y)| \le C|z - y|$ for every $y \in \overline{B}_k$. Thus

$$\begin{aligned}\int_{B_k} |x - \Psi(y)|^{2-n}\, dV_k(y) &= \int_{B_k} |\Psi(z) - \Psi(y)|^{2-n}\, dV_k(y)\\ &\geq C^{2-n} \int_{B_k} |z - y|^{2-n}\, dV_k(y)\\ &= \infty,\end{aligned}$$

where the last equality comes from Exercise 10 of this chapter. The assertion in 9.14 now follows from Fatou's Lemma. ∎

Note that $\overline{B}_1$ is the interval $[-1, 1]$. Thus, any smooth compact arc in Ω is h^∞-removable for Ω provided $n > 2$. Exercise 13 in Chapter 4 and Exercise 12 of this chapter show that compact sets of dimension $n - 1$ are not h^∞-removable.

The Logarithmic Conjugation Theorem

In this section, Ω will denote a connected open subset of $\mathbf{R}^2$. We say that Ω is *finitely connected* if $\mathbf{R}^2 \setminus \Omega$ has finitely many bounded components. Recall that Ω is *simply connected* if $\mathbf{R}^2 \setminus \Omega$ has no bounded components.

If u is the real part of a holomorphic function f on Ω, then the imaginary part of f is called a *harmonic conjugate* of u. When Ω is simply connected, a real-valued harmonic function on Ω always has a harmonic conjugate ([7], Chapter VIII, Theorem 2.2).

The following theorem has been called the logarithmic conjugation theorem because it shows that a real-valued harmonic function on a finitely connected domain has a harmonic conjugate provided that some logarithmic terms are subtracted.

9.15 Logarithmic Conjugation Theorem: *Let Ω be a finitely connected domain. Let $K_1, \dots, K_m$ be the bounded components of $\mathbf{R}^2 \setminus \Omega$, and let $a_j \in K_j$ for $j = 1, \dots, m$. If u is real valued and harmonic on Ω, then there exist f holomorphic on Ω and $b_1, \dots, b_m \in \mathbf{R}$ such that*

$$u(z) = \operatorname{Re} f(z) + b_1 \log|z - a_1| + \cdots + b_m \log|z - a_m|$$

for all $z \in \Omega$.

PROOF: We prove the theorem by induction on m, the number of bounded components in the complement of Ω. To get started, recall that if $m = 0$ then Ω is simply connected and $u = \operatorname{Re} f$ for some function f holomorphic on Ω.

Suppose now that $m > 0$, and that the theorem is true with $m - 1$ in place of m. With Ω as in the statement of the theorem, set $\Omega' = \Omega \cup K_m$, so that Ω' is a finitely connected domain whose complement has $m - 1$ bounded components. Because u is harmonic on $\Omega' \setminus K_m$, 9.7 gives the decomposition $u = v + w$, where v is harmonic on Ω' and w is harmonic on $\mathbf{R}^2 \setminus K_m$, with $\lim_{z\to\infty} w(z) - b\log|z| = 0$ for some constant b.

Because v satisfies the induction hypothesis, we will be done if we can show that

9.16
$$w(z) = \operatorname{Re} g(z) + b\log|z - a_m|$$

for some function g holomorphic on $\mathbf{R}^2 \setminus K_m$.

To verify 9.16, set

$$h(z) = w(z) - b\log|z - a_m|$$

for $z \in \mathbf{R}^2 \setminus K_m$. We easily calculate that $h(z) \to 0$ as $z \to \infty$; thus h extends to be harmonic on $(\mathbf{C} \cup \{\infty\}) \setminus K_m$. Now, $(\mathbf{C} \cup \{\infty\}) \setminus K_m$ can be viewed as a simply connected region on the Riemann sphere. On such a region every real-valued harmonic function has a harmonic conjugate. This gives 9.16, and thus completes the proof of the theorem. ■

As an application of the logarithmic conjugation theorem, we now give a series development for functions harmonic on annuli.

9.17 Theorem: *If u is real valued and harmonic on the annulus $A = \{z \in \mathbf{R}^2 : r_0 < |z| < r_1\}$, then u has a series development of the form*

9.18
$$u(re^{i\theta}) = b\log r + \sum_{k=-\infty}^{\infty} (c_k r^k + \overline{c_{-k}} r^{-k})e^{ik\theta}.$$

The series converges absolutely for each $re^{i\theta} \in A$ and uniformly on compact subsets of A.

PROOF: Use the logarithmic conjugation theorem (9.15) with $\Omega = A$, $K_1 = \{z \in \mathbf{R}^2 : |z| \leq r_0\}$, and $a_1 = 0$, to get

$$u(z) = b \log |z| + \operatorname{Re} f(z)$$

for some holomorphic function f on A. On A, f has a Laurent series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$$

that converges absolutely and uniformly on compact subsets of A. Now,

$$u(z) = b \log |z| + \frac{f(z) + \overline{f(z)}}{2};$$

the series representation 9.18 for u is obtained by setting $z = re^{i\theta}$ and replacing f with its Laurent series. ∎

The series representation 9.18 gives another proof that the averages of u over circles of radius r satisfy the $n = 2$ part of 3.10.

In Chapter 10 we consider the problem of obtaining an analogous series representation for functions harmonic on annular domains in $\mathbf{R}^n$. There, as one might expect, the decomposition theorem (9.6, 9.7) will play an important role.

Additional applications of the logarithmic conjugation theorem may be found in [2].

Exercises

1. Prove Theorem 9.2.

2. Suppose $n > 2$. Given $g \in C_c^2(\mathbf{R}^n)$, show that there exists a unique $u \in C^2(\mathbf{R}^n)$ such that $\Delta u = g$ and $u(x) \to 0$ as $x \to \infty$. What happens when $n = 2$?

3. Prove the decomposition theorem in the $n = 2$ case (9.7).

4. For $\Omega \subset \mathbf{R}^2$, define the operator $\bar{\partial}$ on $C^1(\Omega)$ by

$$\bar{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

Show that if $f \in C^1(\Omega)$, then f is holomorphic on Ω if and only if $\bar{\partial} f = 0$.

5. Show that if $g \in C_c^1(\mathbf{R}^2)$, then

$$g(w) = \frac{1}{\pi}\int_{\mathbf{R}^2} \frac{(\bar{\partial} g)(z)}{w - z}\, dV_2(z)$$

for all $w \in \mathbf{C}$. (*Hint:* Imitate the proof of 9.1, using Green's Theorem instead of Green's identity.)

6. Let $\Omega \subset \mathbf{C}$, let $K \subset \Omega$ be compact, and let f be holomorphic on $\Omega \setminus K$. Using the previous exercise and an argument similar to the proof of the decomposition theorem, prove that f has a unique decomposition of the form $f = g + h$, where g is holomorphic on Ω and h is holomorphic on $\mathbf{C} \setminus K$, with $\lim_{z \to \infty} h(z) = 0$.

7. Prove the Generalized Bôcher Theorem (9.11) when $n > 2$.

8. Show that the Generalized Liouville Theorem (9.10) is a consequence of the Generalized Bôcher Theorem (9.11).

9. If u is a real-valued harmonic function on $\mathbf{R}^n$ such that $u(x)/|x|$ is bounded below for x near ∞, must u be constant?

10. Let $x \in \overline{B}_n$ and let $c \in \mathbf{R}$. Prove that

$$\int_{B_n} |x - y|^c \, dV(y) = \infty$$

if and only if $c \leq -n$.

11. Does the conclusion of Theorem 9.13 remain true if we merely assume that Ψ is continuous on $\overline{B}_k$?

12. Suppose E is a compact subset of $\mathbf{R}^{n-1} \subset \mathbf{R}^n$ with positive $(n-1)$-dimensional Lebesgue measure. Show that

$$u(y) = \int_E |x - y|^{2-n}\, dV_{n-1}(x)$$

defines a function on $\mathbf{R}^n$ that is continuous, bounded, and nonconstant. Show that this function is harmonic on $\mathbf{R}^n \setminus E$.

13. Suppose $\Psi\colon B_k \to B_n$ is a C^1-map, where $1 \le k \le n-2$. Show that if $\Psi(B_k)$ is closed in B_n, then every bounded harmonic function on $B_n \setminus \Psi(B_k)$ extends to a bounded harmonic function on B_n. (Note that Exercise 16 in Chapter 6 is a special case of this exercise.)

14. Every polynomial $p(x, y)$ on $\mathbf{R}^2$ extends to a polynomial $p(z, w)$ on $\mathbf{C}^2$ by replacing x and y with the complex numbers z and w in the expansion of p. Show that if p is a harmonic polynomial on $\mathbf{R}^2$ with real coefficients, then the imaginary part of $2p(z/2, -iz/2)$ is a harmonic conjugate of p.

15. Let $\Omega \subset \mathbf{R}^2$ be finitely connected, and let $K_1, K_2, \dots, K_m$ be the bounded components of $\mathbf{R}^2 \setminus \Omega$. Let $a_j, a'_j \in K_j$. Suppose that u is real valued and harmonic on Ω. Prove that if f, g are holomorphic functions on Ω and $b_j, b'_j \in \mathbf{R}$ satisfy

$$\begin{aligned} u(z) &= \operatorname{Re} f(z) + b_1 \log|z - a_1| + \cdots + b_m \log|z - a_m| \\ &= \operatorname{Re} g(z) + b'_1 \log|z - a'_1| + \cdots + b'_m \log|z - a'_m|, \end{aligned}$$

then $b_j = b'_j$. How are f and g related?

16. Using the logarithmic conjugation theorem (9.15), give another proof of the $n = 2$ case of the Generalized Bôcher Theorem (9.11).

17. Use the series representation 9.18 to show that the Dirichlet problem for an annulus in $\mathbf{R}^2$ is solvable. More precisely, show that if A is an annulus in $\mathbf{R}^2$ and f is continuous on ∂A, then there is a function u harmonic on A and continuous on $\overline{A}$ such that $u|_{\partial A} = f$.

CHAPTER 10

Annular Regions

An *annular region* is a set of the form $\{x \in \mathbf{R}^n : r_0 < |x| < r_1\}$; here $r_0 \in [0, \infty)$ and $r_1 \in (0, \infty]$. Thus an annular region is the region between two concentric spheres, or is a punctured ball, or is the complement of a closed ball, or is $\mathbf{R}^n \setminus \{0\}$.

Laurent Series

If u is harmonic on B, then 5.34 gives the expansion

$$u(x) = \sum_{m=0}^{\infty} p_m(x),$$

where p_m is a homogeneous harmonic polynomial of degree m and the series converges absolutely and uniformly on compact subsets of B. This expansion is reminiscent of the power series expansion for holomorphic functions. We now take up the analogous Laurent series development for harmonic functions on annular regions.

10.1 Laurent Series ($n > 2$): *Suppose u is harmonic on an annular region A. Then there exist unique homogeneous harmonic polynomials p_m and q_m of degree m such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x) + \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+n-2}}$$

on A. The convergence is absolute and uniform on compact subsets of A.

PROOF: Suppose A has inner radius $r_0 \in [0, \infty)$ and outer radius $r_1 \in (0, \infty]$. By the decomposition theorem (9.6) we have $u = v + w$, where v is harmonic on $r_1 B$ and w is harmonic on $(\mathbf{R}^n \cup \{\infty\}) \setminus r_0 \overline{B}$. Because v is harmonic on the ball $r_1 B$, there are homogeneous harmonic polynomials p_m such that

10.2
$$v(x) = \sum_{m=0}^{\infty} p_m(x)$$

on $r_1 B$. The Kelvin transform of $K[w]$ is harmonic on the ball $(1/r_0)B$, and so there are homogeneous harmonic polynomials q_m such that

$$K[w](x) = \sum_{m=0}^{\infty} q_m(x)$$

on $(1/r_0)B$. Applying the Kelvin transform to both sides of this equation, we have

10.3
$$w(x) = \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+n-2}}$$

on $\mathbf{R}^n \setminus r_0 \overline{B}$. Combining the series expansions 10.2 and 10.3, we obtain the desired expansion for u on A. The series 10.2 and 10.3 converge absolutely and uniformly on compact subsets of A, and hence so does the Laurent series expansion of u. Uniqueness of the expansion follows from the uniqueness of the decomposition $u = v + w$ and of the series expansions 10.2 and 10.3. ∎

The preceding proof does not quite work when $n = 2$ because the decomposition theorem takes a different form in that case (see 9.6, 9.7). Exercise 1 of this chapter develops the Laurent series expansion for harmonic functions when $n = 2$.

Isolated Singularities

Suppose $n > 2$, $a \in \Omega$, and u is harmonic on $\Omega \setminus \{a\}$. By Theorem 10.1, there are homogeneous harmonic polynomials p_m and q_m such that

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) + \sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m+n-2}}$$

for x in a deleted neighborhood of a. We call the function

10.4
$$\sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m+n-2}}$$

the *principal part* of u at a and classify the singularity at a accordingly. Specifically, u has a *removable singularity* at a if each term in the principal part is zero; u has a *pole* at a if the principal part is a finite sum of nonzero terms; u has an *essential singularity* at a if the principal part has infinitely many nonzero terms.

If u has a pole at a, with principal part given by 10.4, and M is the largest integer such that $q_M \neq 0$, then we say that the pole has *order* $M + n - 2$. For example, if α is a multi-index, then $D^\alpha |x|^{2-n}$ has a pole of order $|\alpha| + n - 2$ at 0. Theorem 10.5(b) below shows why the order of a pole has been defined in this manner. We call a pole of order $n - 2$ a *fundamental pole* (because the principal part is then a multiple of the fundamental solution defined in Chapter 9).

10.5 Theorem ($n > 2$): *If u is harmonic with an isolated singularity at a, then u has*

(a) *a removable singularity at a if and only if*

$$\lim_{x \to a} |x-a|^{n-2} |u(x)| = 0;$$

(b) *a pole at a of order $M + n - 2$ if and only if*

$$0 < \limsup_{x \to a} |x-a|^{M+n-2} |u(x)| < \infty;$$

(c) *an essential singularity at a if and only if*

$$\limsup_{x \to a} |x-a|^{N} |u(x)| = \infty$$

for every positive integer N.

PROOF: The proof of (a) follows from Exercise 2(a) in Chapter 2.

For the remainder of the proof, we assume that u has principal part at a given by $w(x) = \sum_{m=0}^{\infty} q_m(x-a)/|x-a|^{2m+n-2}$.

To prove (b), first suppose that u has a pole at a of order $M+n-2$. Then the homogeneity of each q_m implies that

$$\limsup_{x\to a} |x-a|^{M+n-2}|u(x)| = \sup_S |q_M|.$$

The right side of this equation is positive and finite, and hence so is the left side, proving one direction of (b).

Conversely, suppose $0 < \limsup_{x\to a} |x-a|^{M+n-2}|u(x)| < \infty$. Then there is a constant $C < \infty$ such that $|w(a+r\zeta)| \le C/r^{M+n-2}$ for small $r > 0$ and $\zeta \in S$. Let j be an integer with $j > M$. Then

$$\begin{aligned}\frac{\int_S |q_j(\zeta)|^2\, d\sigma(\zeta)}{r^{2j+2n-4}} &\le \sum_{m=0}^{\infty} \frac{\int_S |q_m(\zeta)|^2\, d\sigma(\zeta)}{r^{2m+2n-4}}\\ &= \int_S |w(a+r\zeta)|^2\, d\sigma(\zeta)\\ &\le \frac{C^2}{r^{2M+2n-4}}\end{aligned}$$

for small $r > 0$; here we have used the orthogonality of spherical harmonics of different degree (Proposition 5.9). Letting $r \to 0$, we get $\int_S |q_j|^2\, d\sigma = 0$, so that $q_j \equiv 0$. Thus u has a pole at a of order at most $M+n-2$. Because $\limsup_{x\to a} |x-a|^{M+n-2}|u(x)|$ is positive, the order of the pole is at least $M+n-2$, completing the proof of (b).

To prove (c), first suppose that $\limsup_{x\to a} |x-a|^N|u(x)| = \infty$ for every positive integer N. By (a) and (b), u can have neither a removable singularity nor a pole at a, and thus u has an essential singularity at a.

To prove the other direction of (c), suppose there is a positive integer N such that $\limsup_{x\to a} |x-a|^N|u(x)| < \infty$. By the argument used in proving (b), this implies that $q_j \equiv 0$ for all sufficiently large j. Thus u does not have an essential singularity at a, completing the proof of (c). ■

The analogue of the theorem above for $n = 2$, along with the appropriate definitions, is given in Exercise 2 of this chapter.

Recall that Picard's Theorem states if f is a holomorphic function with essential singularity at a, then f assumes all complex values, with

one possible exception, infinitely often on every deleted neighborhood of a. Picard's Theorem has the following analogue for real-valued harmonic functions.

10.6 Theorem: *Let u be a real-valued harmonic function with either an essential singularity or a pole of order greater than $n-2$ at $a \in \mathbf{R}^n$. Then u assumes every real value infinitely often near a.*

PROOF: By Bôcher's Theorem (3.9), u cannot be bounded above or below on any deleted neighborhood of a. Thus, for every small $r > 0$, the connected set $u(B(a,r) \setminus \{a\})$ must be all of $\mathbf{R}$. This implies that u assumes every real value infinitely often near a. ∎

There is no analogue of Theorem 10.6 for complex-valued harmonic functions.

The Residue Theorem

Suppose $u \in C^2(\Omega)$. Then u is harmonic on Ω if and only if

$$\int_{\partial B(a,r)} D_{\mathbf{n}} u \, ds = 0$$

for every closed ball $\overline{B}(a,r) \subset \Omega$; as usual, $D_{\mathbf{n}}$ denotes the derivative with respect to the outward normal $\mathbf{n}$ and ds denotes (unnormalized) surface-area measure. Proof: apply Green's identity (1.1) with $v \equiv 1$ to small closed balls contained in Ω. We can think of this result as an analogue of Morera's Theorem for holomorphic functions.

Integrating the normal derivative over the boundary also yields a "residue theorem" of sorts. Suppose $n > 2$ and the harmonic function u has an isolated singularity at a, with Laurent series expansion at a given by

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) + \sum_{m=0}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m+n-2}}.$$

We call the constant q_0 the *residue* of u at a, and write $\operatorname{Res}(u,a) = q_0$. The following proposition and theorem justify this terminology.

10.7 Proposition ($n > 2$): *If u is harmonic on $\overline{B}(a,r) \setminus \{a\}$, then*

$$\operatorname{Res}(u,a) = \frac{1}{(2-n)nV(B)} \int_{\partial B(a,r)} D_{\mathbf{n}} u \, ds.$$

PROOF: Without loss of generality, assume $a = 0$. Suppose

$$u(x) = \sum_{m=0}^{\infty} p_m(x) + \sum_{m=0}^{\infty} \frac{q_m(x)}{|x|^{2m+n-2}}$$

is the Laurent expansion of u about 0. The first sum is harmonic on $\overline{B}(0,r)$; hence, the integral of its normal derivative over $\partial B(0,r)$ is zero. The integral of the normal derivative of the second sum equals

$$q_0 r^{1-n}(2-n) \int_{\partial B(0,r)} ds + \sum_{m=1}^{\infty} (2-n-m) r^{1-n-2m} \int_{\partial B(0,r)} q_m \, ds.$$

The value of the first integral is $q_0 nV(B)(2-n)$; all other integrals vanish by the mean-value property. ∎

10.8 Residue Theorem ($n > 2$): *Suppose Ω is a bounded open set with smooth boundary. Let $a_1, \dots, a_k$ be distinct points in Ω. If u is harmonic on $\overline{\Omega} \setminus \{a_1, \dots, a_k\}$, then*

$$\int_{\partial\Omega} D_{\mathbf{n}} u \, ds = (2-n)nV(B) \sum_{j=1}^{k} \operatorname{Res}(u, a_j).$$

PROOF: Choose $r > 0$ so that $\overline{B}(a_1,r), \dots, \overline{B}(a_k,r)$ are pairwise disjoint and all contained in Ω. Set $\omega = \Omega \setminus (\bigcup_{j=1}^{k} \overline{B}(a_j,r))$. Then

$$\int_{\partial\omega} D_{\mathbf{n}} u \, ds = 0$$

by Green's identity (1.3). Hence

$$\begin{aligned} \int_{\partial\Omega} D_{\mathbf{n}} u \, ds &= -\sum_{j=1}^{k} \int_{\partial B(a_j,r)} D_{\mathbf{n}} u \, ds \\ &= (2-n)nV(B) \sum_{j=1}^{k} \operatorname{Res}(u, a_j) \end{aligned}$$

(note that $\mathbf{n}$ points toward a_j on $\partial B(a_j,r)$). ∎

See Exercise 8 of this chapter for statement of the residue theorem when $n = 2$.

The Poisson Kernel for Annular Regions

Let A be a bounded annular region. If f is a continuous function on ∂A, does f have a continuous extension to $\overline{A}$ that is harmonic on A? In this section we will see that this question, called the Dirichlet problem for A, has an affirmative answer if the inner radius of A is positive. In fact, we will find a Poisson-integral type formula for the solution. (In the next chapter, we show that the Dirichlet problem is solvable on a much wider class of domains, although in the more general context we will not have an explicit integral formula for the solution.)

Fix $r_0 \in (0,1)$. Throughout this section, we assume that A is the annular region $\{x \in \mathbf{R}^n : r_0 < |x| < 1\}$. This is no loss of generality because dilations preserve harmonic functions.

To discover the formula for solving the Dirichlet problem for A, we begin with a special case. Suppose $g \in \mathcal{H}_m(S)$ for some $m \geq 0$. Consider the problem of finding a continuous function u on $\overline{A}$ that is harmonic on A, with $u = g$ on S and $u = 0$ on $r_0 S$. We first extend g to a harmonic homogeneous polynomial of degree m (which we also denote by g). The Kelvin transform of g is then harmonic on $\mathbf{R}^n \setminus \{0\}$; the homogeneity of g shows that $K[g](x) = g(x)/|x|^{2m+n-2}$. Thus the function u defined by

$$u(x) = \frac{1 - (r_0/|x|)^{2m+n-2}}{1 - r_0{}^{2m+n-2}} g(x)$$

solves the Dirichlet problem in this special case.

Let us define

10.9
$$b_m(x) = \frac{1 - (r_0/|x|)^{2m+n-2}}{1 - r_0{}^{2m+n-2}},$$

so that $u(x) = b_m(x)g(x)$, where u is the function displayed above. Recall that integration against the zonal harmonic $Z_m(x, \zeta)$ reproduces the values of functions in $\mathcal{H}_m(\mathbf{R}^n)$ (5.30). Thus we can rewrite our formula for the solution u as follows:

$$u(x) = b_m(x) \int_S g(\zeta) Z_m(x, \zeta)\, d\sigma(\zeta).$$

Going one step further, if $g = \sum_{m=0}^{M} g_m$, where $g_m \in \mathcal{H}_m(S)$, then adding the solutions obtained for each g_m in the previous paragraph solves the Dirichlet problem for A with boundary data g on S and 0 on $r_0 S$. Explicitly,

$$u(x) = \int_S g(\zeta)\Big(\sum_{m=0}^{M} b_m(x) Z_m(x,\zeta)\Big)\, d\sigma(\zeta).$$

Note that for each $\zeta \in S$, the function $x \mapsto b_m(x) Z_m(x,\zeta)$ is harmonic on A (because it equals a constant times $Z_m(x,\zeta)$ plus a constant times $K[Z_m(\cdot,\zeta)](x)$).

Any polynomial restricted to S is the sum of spherical harmonics (from Theorem 5.7). Furthermore, the set of polynomials is dense in $C(S)$ by the Stone-Weierstrass Theorem (see [14], Theorem 7.33). Suppose, then, that g is an arbitrary continuous function on S. To find a continuous function u on $\overline{A}$ that is harmonic on A, with $u = g$ on S and $u = 0$ on $r_0 S$, the results above suggest that we try

$$u(x) = \int_S g(\zeta) P_A(x,\zeta)\, d\sigma(\zeta),$$

where

10.10 $$P_A(x,\zeta) = \sum_{m=0}^{\infty} b_m(x) Z_m(x,\zeta).$$

Note that $0 < b_m(x) < 1$ for $x \in A$. Thus the last series converges absolutely and uniformly on $K \times S$ for every compact $K \subset A$, as in the proof of Theorem 5.33. In particular, for each $\zeta \in S$, the function $P_A(\cdot,\zeta)$ is harmonic on A.

We handle the Dirichlet problem for A with boundary data h on $r_0 S$ and 0 on S in a similar manner. Thus a process like the one above suggests that the solution u is given by the formula

$$u(x) = \int_S h(r_0\zeta) P_A(x, r_0\zeta)\, d\sigma(\zeta),$$

where

10.11 $$P_A(x, r_0\zeta) = \sum_{m=0}^{\infty} c_m(x) Z_m(x,\zeta)$$

and

10.12 $$c_m(x) = |x|^{-m}(r_0/|x|)^{m+n-2}\frac{1-|x|^{2m+n-2}}{1-r_0{}^{2m+n-2}}.$$

Note that $|c_m(x)Z_m(x,\zeta)| < (r_0/|x|)^{m+n-2}|Z_m(x/|x|,\zeta)|$ for $x \in A$, so the infinite sum in 10.11 converges absolutely and uniformly on $K \times S$ for every compact $K \subset A$, as in the proof of Theorem 5.33. In particular, for each $\zeta \in S$, the function $P_A(\cdot, r_0\zeta)$ is harmonic on A.

Having approached the Dirichlet problem for A "one sphere at a time", we easily guess what to do for an arbitrary $f \in C(\partial A)$—we simply add the two candidate solutions obtained above.

We now make the formal definitions. For $n > 2$, P_A is the function on $A \times \partial A$ defined by 10.9-10.12. (For $n = 2$ and $m = 0$, the terms $b_0(x)$ and $c_0(x)$ must be replaced by appropriate modifications of $\log|x|$; Exercise 10 of this chapter asks the reader to make the necessary adjustments.) For $f \in C(\partial A)$, the Poisson integral of f, denoted $P_A[f]$, is the function on A defined by

$$P_A[f](x) = \int_S f(\zeta)P_A(x,\zeta)\,d\sigma(\zeta) + \int_S f(r_0\zeta)P_A(x, r_0\zeta)\,d\sigma(\zeta).$$

We have already done most of the work needed to show that $P_A[f]$ solves the Dirichlet problem for f.

10.13 Theorem ($n > 2$): *Suppose f is continuous on ∂A. Define u on $\overline{A}$ by*

$$u(x) = \begin{cases} P_A[f](x) & \text{if } x \in A \\ f(x) & \text{if } x \in \partial A. \end{cases}$$

Then u is continuous on $\overline{A}$ and harmonic on A.

PROOF: The function $P_A[f]$ is the sum of two harmonic functions, and hence is harmonic.

To complete the proof, we need only show that u is continuous on $\overline{A}$. The discussion above shows that u is continuous on $\overline{A}$ in the case where $f|_S$ and $f(r_0\cdot)|_S$ are both finite sums of spherical harmonics. By Theorem 5.7 and the Stone-Weierstrass Theorem (see [14], Theorem 7.33), such functions are dense in $C(\partial A)$. For the general $f \in C(\partial A)$, we approximate f uniformly on ∂A with functions $f_1, f_2, \ldots$ from this dense

subspace; the corresponding solutions $u_1, u_2, \dots$ then converge uniformly to u on $\overline{A}$ by the maximum principle, proving that u is continuous on $\overline{A}$. ■

The hypothesis that r_0 be greater than 0 is needed to solve the Dirichlet problem for the annular region A. For example, there is no function u harmonic on the punctured ball $B \setminus \{0\}$, with u continuous on $\overline{B}$, satisfying $u = 1$ on S and $u(0) = 0$: if there were such a function, then it would be bounded and harmonic on $B \setminus \{0\}$, hence would extend harmonically to B (by 2.3), contradicting the minimum principle.

The results in this section are used by the software described in Appendix B to solve the Dirichlet problem for annular regions. For example, the software computes that the harmonic function on the annular region $\{x \in \mathbf{R}^3 : 2 \le |x| \le 3\}$ that equals ${x_1}^2$ when $|x| = 2$ and equals $x_1x_2x_3$ when $|x| = 3$ is

$$-\frac{8}{3} - \frac{2592}{211|x|^3} + \frac{8}{|x|} + \frac{32|x|^2}{633} - \frac{32{x_1}^2}{211} + \frac{7776{x_1}^2}{211|x|^5} + \frac{2187x_1x_2x_3}{2059} - \frac{279936x_1x_2x_3}{2059|x|^7}.$$

Exercises

1. Suppose u is harmonic on an annular region A in $\mathbf{R}^2$. Show that there exist $p_m, q_m \in \mathcal{H}_m(\mathbf{R}^2)$ such that

$$u(x) = \sum_{m=0}^{\infty} p_m(x) + q_0 \log|x| + \sum_{m=1}^{\infty} \frac{q_m(x)}{|x|^{2m}}$$

on A. Show also that the series converges absolutely and uniformly on compact subsets of A.

2. Suppose u is a harmonic function with an isolated singularity at $a \in \mathbf{R}^2$. The *principal part* of u at a is defined to be

$$q_0 \log|x-a| + \sum_{m=1}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m}},$$

where u has been expanded about a as in Exercise 1. We say that u has a *fundamental pole* at a if the principal part is a nonzero multiple of $\log|x|$. We say that u has a *pole* at a of *order* M if there is a largest positive integer M such that $q_M \neq 0$. We say that u has an *essential singularity* at a if the principal part has infinitely many nonzero terms. Prove that u has

(a) a removable singularity at a if and only if

$$\lim_{x \to a} \frac{u(x)}{\log|x-a|} = 0;$$

(b) a fundamental pole at a if and only if

$$0 < \lim_{x \to a} \left| \frac{u(x)}{\log|x-a|} \right| < \infty;$$

(c) a pole at a of order M if and only if

$$0 < \limsup_{x \to a} |x-a|^M |u(x)| < \infty;$$

(d) an essential singularity at a if and only if

$$\limsup_{x \to a} |x-a|^N |u(x)| = \infty$$

for every positive integer N.

3. Give an example of a harmonic function of n variables, $n > 2$, that has an essential singularity at 0.

4. Let u be a real-valued harmonic function with an isolated singularity at $a \in \mathbf{R}^n$. Show that u has a fundamental pole at a if and only if

$$\lim_{x \to a} |u(x)| = \infty.$$

5. Suppose $n > 2$ and u is a harmonic function with an isolated singularity at a. Prove that

$$\lim_{x \to a} |x - a|^{n-2} u(x)$$

exists (as a complex number) if and only if u has either a removable singularity or a fundamental pole at a.

6. *Singularities at* ∞*:* Suppose u is harmonic on a deleted neighborhood of ∞. The singularity of u at ∞ is classified using the Laurent expansion of the Kelvin transform $K[u]$ at 0; for example, if the Laurent expansion of $K[u]$ at 0 has vanishing principal part, then we say u has a *removable singularity* at ∞.

   (a) Show that u has an essential singularity at ∞ if and only if

   $$\limsup_{x \to \infty} \frac{|u(x)|}{|x|^M} = \infty$$

   for every positive integer M.

   (b) Find growth estimates that characterize the other types of singularities at ∞.

7. (a) Identify those functions that are harmonic on $\mathbf{R}^n$, $n > 2$, with fundamental pole at ∞.

   (b) Identify those functions that are harmonic on $\mathbf{R}^2$ with fundamental pole at ∞.

8. Suppose $n = 2$ and the harmonic function u has an isolated singularity at $a \in \mathbf{R}^2$, with Laurent series expansion

$$u(x) = \sum_{m=0}^{\infty} p_m(x-a) + q_0 \log|x-a| + \sum_{m=1}^{\infty} \frac{q_m(x-a)}{|x-a|^{2m}}.$$

We say that the constant q_0 is the *residue* of u at a and write $\operatorname{Res}(u,a) = q_0$. Prove that if u is harmonic on $\overline{B}(a,r) \setminus \{a\}$, then

$$\operatorname{Res}(u,a) = \frac{1}{2\pi} \int_{\partial B(a,r)} D_{\mathbf{n}} u \, ds.$$

Also prove an analogue of the residue theorem (10.8) for the case $n = 2$.

9. Show how formulas 10.11 and 10.12 are derived.

10. Find the correct replacements for 10.9 and 10.12 when $n = 2$ and $m = 0$, and use this to solve the Dirichlet problem for annular regions in the plane.

11. Let $0 < r_0 < 1$ and let $A = \{x \in \mathbf{R}^n : r_0 < |x| < 1\}$. Let p, q be polynomials on $\mathbf{R}^n$, and let f be the function on ∂A that equals p on $r_0 S$ and equals q on S. Prove that $P_A[f]$ extends to a function that is harmonic on $\mathbf{R}^n \setminus \{0\}$.

12. *Generalized Annular Dirichlet Problem:* Suppose that A is the annulus $\{x \in \mathbf{R}^n : r_0 < |x| < r_1\}$, where $0 < r_0 < r_1 < \infty$. Prove that if f, g, h are polynomials on $\mathbf{R}^n$, then there is a function $u \in C(\overline{A})$ such that $u = f$ on $r_0 S$, $u = g$ on $r_1 S$, and $\Delta u = h$ on A. Show that if $n > 2$, then u is a finite sum of functions of the form $p(x)/|x|^m$, where p is a polynomial on $\mathbf{R}^n$ and m is a nonnegative integer. (The software described in Appendix B can find u explicitly.)

CHAPTER 11

The Dirichlet Problem and Boundary Behavior

In this chapter we construct harmonic functions on Ω that behave in a prescribed manner near $\partial\Omega$. Here we are interested in general domains $\Omega \subset \mathbf{R}^n$; the techniques we developed for the special domains B and H will not be available. Most of this chapter will concern the Dirichlet problem. In the last section, however, we will study a different kind of boundary behavior problem—the construction of harmonic functions on Ω that cannot be extended harmonically across any part of $\partial\Omega$.

The Dirichlet Problem

If f is a continuous function on $\partial\Omega$, does f have a continuous extension to $\overline{\Omega}$ that is harmonic on Ω? This is the *Dirichlet problem* for Ω with boundary data f. If the answer is affirmative for all continuous f on $\partial\Omega$, we say that the Dirichlet problem for Ω is *solvable*. Recall that the Dirichlet problem is solvable for B (Theorem 1.17) and for the region between two concentric spheres (Theorem 10.13), but not for the punctured ball $B \setminus \{0\}$ (see the remark following the proof of 10.13). The Dirichlet problem is sometimes referred to as "the first boundary value problem of potential theory". The search for its solution led to the development of much of harmonic function theory.

We will obtain a necessary and sufficient condition for the Dirichlet problem to be solvable for bounded Ω. Although the condition is not

entirely satisfactory, it leads in many cases to easily verified geometric criteria that imply the Dirichlet problem is solvable. We will see, for example, that the Dirichlet problem is solvable for bounded Ω whenever Ω is convex or whenever $\partial\Omega$ is "smooth".

Note that when Ω is bounded, the maximum principle (1.9) implies that if a solution to the Dirichlet problem exists, then it is unique.

Subharmonic Functions

We follow the so-called Perron approach in solving the Dirichlet problem. This ingenious method constructs a solution as the supremum of a family of subharmonic functions. In this book, we call a real-valued function u *subharmonic* on Ω provided u is continuous on Ω and u satisfies the *submean-value property* on Ω. The latter requirement is that for each $a \in \Omega$, there exists a closed ball $\overline{B}(a, R) \subset \Omega$ such that

11.1
$$u(a) \le \int_S u(a + r\zeta)\, d\sigma(\zeta)$$

whenever $0 < r \le R$. Note that we are not requiring 11.1 to hold for all $r < d(a, \partial\Omega)$. (But see Exercise 5 of this chapter.)

Obviously every real-valued harmonic function on Ω is subharmonic on Ω. A finite sum of subharmonic functions is subharmonic, as is any positive scalar multiple of a subharmonic function. In Exercise 8 of this chapter we ask the reader to prove that a real-valued $u \in C^2(\Omega)$ is subharmonic on Ω if and only if $\Delta u \ge 0$ on Ω. Thus $u(x) = |x|^2$ is a subharmonic function on $\mathbf{R}^n$ that is not harmonic. This example shows that subharmonic functions do not satisfy the minimum principle. They do, however, satisfy the maximum principle.

11.2 Theorem: *Suppose Ω is connected and u is subharmonic on Ω. If u has a maximum in Ω, then u is constant.*

PROOF: Suppose u attains a maximum at $a \in \Omega$. Choose a closed ball $\overline{B}(a, R) \subset \Omega$ as in 11.1. We have $u \le u(a)$ on $B(a, R)$. If u were less than $u(a)$ at any point of $B(a, R)$, then the continuity of u would show that 11.1 fails for some $r < R$. Thus $u \equiv u(a)$ on $B(a, R)$. The set where u attains its maximum is therefore an open subset of Ω. Because

this set is also closed in Ω, it must be all of Ω by the connectivity of Ω, proving that u is constant on Ω. ∎

The following theorem indicates another sense in which subharmonic functions are "sub"-harmonic.

11.3 Theorem: *Let Ω be bounded. Suppose u and v are continuous on $\overline{\Omega}$, u is subharmonic on Ω, and v is harmonic on Ω. If $u \le v$ on $\partial\Omega$, then $u \le v$ on $\overline{\Omega}$.*

PROOF: Suppose $u - v \le 0$ on $\partial\Omega$. Because $u - v$ is subharmonic on Ω, the maximum principle (11.2) shows that $u - v \le 0$ throughout Ω. ∎

The proof of the next result follows easily from the definition of subharmonic functions, and we leave it to the reader.

11.4 Proposition: *If u_1 and u_2 are subharmonic functions on Ω, then $\max\{u_1, u_2\}$ is subharmonic on Ω.*

Although Proposition 11.4 is easy, it indicates why subharmonic functions are useful—they can be "bent" in ways that harmonic functions simply would not tolerate. The next theorem is a more sophisticated bending result.

11.5 Theorem: *Suppose that u is subharmonic on Ω and $\overline{B}(a,R) \subset \Omega$. Let w be the function that on $\Omega \setminus B(a,R)$ equals $u|_{\Omega\setminus B(a,R)}$ and that on $B(a,R)$ equals the solution to the Dirichlet problem for $B(a,R)$ with boundary data $u|_{\partial B(a,R)}$. Then w is subharmonic on Ω and $u \le w$.*

PROOF: The inequality $u \le w$ on Ω follows from Theorem 11.3. The continuity of w on Ω is clear.

To verify that w satisfies the submean-value property on Ω, let $b \in \Omega$. If $b \in B(a,R)$, then the harmonicity of w near b implies that $w(b) = \int_S w(b + r\zeta)\, d\sigma(\zeta)$ for all sufficiently small r. If $b \notin B(a,R)$, then $u(b) = w(b)$. The subharmonicity of u, coupled with the inequality $u \le w$ on Ω, then implies that $w(b) \le \int_S w(b + r\zeta)\, d\sigma(\zeta)$ for all sufficiently small r. Thus w is subharmonic on Ω. ∎

We call the function w defined in Theorem 11.5 the *Poisson modification* of u for $B(a,R)$.

The Perron Construction

In this and the next section, Ω will denote a bounded open subset of $\mathbf{R}^n$ and f will denote a continuous real-valued function on $\partial\Omega$. (Note that the Dirichlet problem is no less general if one assumes the boundary data to be real.) We define S_f to be the family of real-valued, continuous functions u on $\overline{\Omega}$ that are subharmonic on Ω and satisfy $u \le f$ on $\partial\Omega$. The collection S_f is often called the *Perron family* for f. Note that S_f is never empty—it contains the constant function $u(x) = m$, where m is the minimum value of f on $\partial\Omega$. (We are already using the boundedness of Ω.)

The two bending processes mentioned in the last section preserve the family S_f. Specifically, if u_1, u_2 belong to S_f, then so does the function $\max\{u_1, u_2\}$; if $u \in S_f$ and $\overline{B}(a,R) \subset \Omega$, then the Poisson modification of u for $B(a,R)$ belongs to S_f.

Perron's candidate solution for the Dirichlet problem with boundary data f is the function defined on $\overline{\Omega}$ by

$$\mathcal{P}[f](x) = \sup\{u(x) : u \in S_f\}.$$

We call $\mathcal{P}[f]$ the *Perron function* for f. Note that $m \le \mathcal{P}[f] \le M$ on $\overline{\Omega}$, where m and M are the minimum and maximum values of f on $\partial\Omega$. Note also that $\mathcal{P}[f] \le f$ on $\partial\Omega$.

To motivate the definition of $\mathcal{P}[f]$, suppose that v solves the Dirichlet problem for Ω with boundary data f. Then $v \in S_f$, so that $v \le \mathcal{P}[f]$ on $\overline{\Omega}$. On the other hand, Theorem 11.3 shows that every function in S_f is bounded above on $\overline{\Omega}$ by v, so that $\mathcal{P}[f] \le v$ on $\overline{\Omega}$. In other words, if there is a solution, it must be $\mathcal{P}[f]$.

Remarkably, even though $\mathcal{P}[f]$ may not be a solution, it is always harmonic.

11.6 Theorem: *$\mathcal{P}[f]$ is harmonic on Ω.*

PROOF: Let $\overline{B}(a,R) \subset \Omega$. It suffices to show that $\mathcal{P}[f]$ is harmonic on $B(a,R)$.

Choose a sequence (u_k) in S_f such that $u_k(a) \to \mathcal{P}[f](a)$. Replacing u_k by the Poisson modification of $\max\{u_1, \dots, u_k\}$ for $B(a,R)$, we obtain a sequence in S_f that increases on Ω and whose terms are harmonic on $B(a,R)$. Denoting this new sequence by (u_k) as well, we still

have $u_k(a) \to \mathcal{P}[f](a) < \infty$. By Harnack's Principle (3.8), (u_k) converges uniformly on compact subsets of $B(a,R)$ to a function u harmonic on $B(a,R)$. The proof will be completed by showing $\mathcal{P}[f] = u$ on $B(a,R)$.

We clearly have $u \le \mathcal{P}[f]$ on $B(a,R)$. To prove the reverse inequality, let $v \in S_f$; we need to show that $v \le u$ on $B(a,R)$. Let v_k denote the Poisson modification of $\max\{u_k, v\}$ for $B(a,R)$. Because $u(a) = \mathcal{P}[f](a)$ and $v_k \in S_f$, we have $v_k(a) \le u(a)$ for all k. Furthermore, v_k is harmonic on $B(a,R)$ and $\max\{u_k, v\} \le v_k$ on $\overline{B}(a,R)$ by subharmonicity. Thus for positive $r < R$,

$$\begin{aligned} u(a) &\ge v_k(a) \\ &= \int_S v_k(a + r\zeta)\, d\sigma(\zeta) \\ &\ge \int_S [\max\{u_k, v\}](a + r\zeta)\, d\sigma(\zeta). \end{aligned}$$

Letting $k \to \infty$, we see that the mean-value property of u on $B(a,R)$ gives

$$\int_S u(a + r\zeta)\, d\sigma(\zeta) \ge \int_S [\max\{u, v\}](a + r\zeta)\, d\sigma(\zeta).$$

It follows that $v \le u$ on $B(a,R)$, proving that $\mathcal{P}[f] \le u$ on $B(a,R)$, as desired. ∎

Barrier Functions and Geometric Criteria for Solvability

Let $\zeta \in \partial\Omega$. We call a continuous real-valued function u on $\overline{\Omega}$ a *barrier function* for Ω at ζ provided that u is subharmonic on Ω, $u < 0$ on $\overline{\Omega} \setminus \{\zeta\}$, and $u(\zeta) = 0$. When such a u exists, we say that Ω has a *barrier* at ζ. (After Theorems 11.11 and 11.16, the reader may concede that the term "barrier" is apt. Poincaré introduced barrier functions into the study of the Dirichlet problem; Lebesgue coined the term "barrier" and generalized the notion.)

11.7 Theorem: *If Ω has a barrier at $\zeta \in \partial\Omega$, then $\mathcal{P}[f](x) \to f(\zeta)$ as $x \to \zeta$ within $\overline{\Omega}$.*

PROOF: Suppose u is a barrier function for Ω at $\zeta \in \partial\Omega$. Let $\varepsilon > 0$. By the continuity of f on $\partial\Omega$, we may choose a ball $B(\zeta, r)$ such that $f(\zeta) - \varepsilon < f < f(\zeta) + \varepsilon$ on $\partial\Omega \cap B(\zeta, r)$. Because u is negative and continuous on the compact set $\overline{\Omega} \setminus B(\zeta, r)$, there exists a positive constant C such that

$$f(\zeta) - \varepsilon + Cu < f < f(\zeta) + \varepsilon - Cu \tag{11.8}$$

on $\partial\Omega \setminus B(\zeta, r)$. The nonpositivity of u then shows that 11.8 is valid everywhere on $\partial\Omega$.

We claim that

$$f(\zeta) - \varepsilon + Cu \le \mathcal{P}[f] \le f(\zeta) + \varepsilon - Cu \tag{11.9}$$

on $\overline{\Omega}$. The first inequality in 11.9 holds because $f(\zeta) - \varepsilon + Cu \in S_f$. For the other inequality, let $v \in S_f$. Then $v \le f$ on $\partial\Omega$, and therefore $v + Cu < f(\zeta) + \varepsilon$ on $\partial\Omega$ by 11.8. Theorem 11.3 then shows that $v + Cu < f(\zeta) + \varepsilon$ on $\overline{\Omega}$, from which the inequality on the right of 11.9 follows.

Because $u(\zeta) = 0$ and ε is arbitrary, the continuity of u and 11.9 give us the desired convergence of $\mathcal{P}[f](x)$ to $f(\zeta)$ as $x \to \zeta$ within $\overline{\Omega}$. ∎

11.10 Theorem: *The Dirichlet problem for bounded Ω is solvable if and only if Ω has a barrier at each $\zeta \in \partial\Omega$.*

PROOF: Suppose that the Dirichlet problem for Ω is solvable and $\zeta \in \partial\Omega$. The function f defined on $\partial\Omega$ by $f(x) = -|x - \zeta|$ is continuous on $\partial\Omega$; the solution to the Dirichlet problem for Ω with boundary data f is then a barrier function for Ω at ζ.

Conversely, suppose that each point of $\partial\Omega$ has a barrier function. Theorems 11.6 and 11.7 then show that $\mathcal{P}[f]$ solves the Dirichlet problem with boundary data f whenever f is continuous and real valued on $\partial\Omega$. ∎

Theorem 11.10 reduces the Dirichlet problem to a local boundary behavior question that we call the *barrier problem.* Have we made

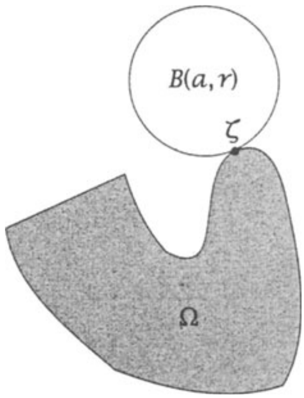
progress, or have we merely traded one boundary behavior problem for another? We will see that the barrier problem can be solved in many cases of interest. For example, the next result will enable us to prove that the Dirichlet problem is solvable for any bounded convex domain.

11.11 External Ball Condition: *If $\zeta \in \partial\Omega$ belongs to a closed ball contained in the complement of Ω, then Ω has a barrier at ζ.*

PROOF: Suppose $\overline{B}(a,r)$ is a closed ball in the complement of Ω such that $\partial B(a,r) \cap \partial\Omega = \{\zeta\}$. Define u on $\mathbf{R}^n \setminus \{a\}$ by

$$u(x) = \begin{cases} \log r - \log|x-a| & \text{if } n = 2 \\ |x-a|^{2-n} - r^{2-n} & \text{if } n > 2. \end{cases}$$

Then u is a barrier for Ω at ζ. ■



The external ball condition.

Consider now the case where Ω is bounded and convex. Each $\zeta \in \partial\Omega$ then belongs to a closed half-space contained in the complement of Ω, which implies that the external ball condition is satisfied at each $\zeta \in \partial\Omega$. By 11.11, Ω has a barrier at every point in its boundary. The following corollary is thus a consequence of Theorem 11.10.

11.12 Corollary: *If Ω is bounded and convex, then the Dirichlet problem for Ω is solvable.*

Theorem 11.11 also enables us to prove that the Dirichlet problem is solvable for bounded Ω when $\partial\Omega$ is sufficiently smooth. To make this more precise, let us say that Ω has *C^k-boundary* if for every $\zeta \in \partial\Omega$ there exists a neighborhood ω of ζ and a real-valued $\varphi \in C^k(\omega)$ satisfying the following conditions:

(a) $\Omega \cap \omega = \{x \in \omega : \varphi(x) < 0\}$;

(b) $\partial\Omega \cap \omega = \{x \in \omega : \varphi(x) = 0\}$;

(c) $\nabla\varphi(\zeta) \neq 0$ for every $\zeta \in \partial\Omega \cap \omega$.

Here k is any positive integer. The function φ is called a *local defining function* for Ω.

Assume now that Ω has C^2-boundary. For simplicity, suppose that $0 \in \partial\Omega$ and that the tangent space of $\partial\Omega$ at 0 is $\mathbf{R}^{n-1} \times \{0\}$. Then near 0, $\partial\Omega$ is the graph of a C^2-function ψ, where ψ is defined near $0 \in \mathbf{R}^{n-1}$ and $\nabla\psi(0) = 0$; this follows easily from the implicit function theorem. Because $\nabla\psi(0) = 0$, we have $|\psi(x)| = O(|x|^2)$ as $x \to 0$ (by Taylor's Theorem). This implies (we leave the details to the reader) that Ω satisfies the external ball condition at 0. The preceding argument, after a translation and rotation, applies to any boundary point of Ω. From Theorems 11.10 and 11.11 we thus obtain the following result.

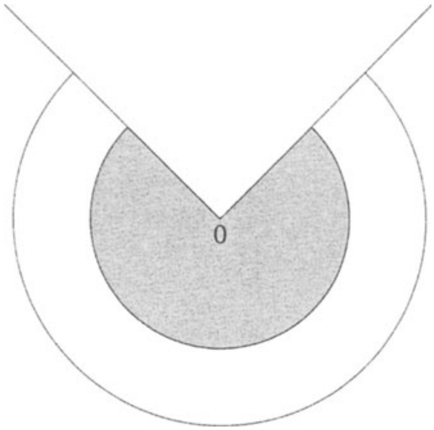
11.13 Corollary: *If Ω is bounded and has C^2-boundary, then the Dirichlet problem for Ω is solvable.*

A domain with C^1-boundary need not satisfy the external ball condition (Exercise 16(a) of this chapter). We now take up a condition on $\partial\Omega$ that covers the C^1-case as well as many "nonsmooth" cases. The prototype for this more-general Ω is the domain $B \setminus \overline{\Gamma}_\alpha(0)$, where $\Gamma_\alpha(0)$ is the cone defined in Chapter 2; see the following diagram, where $B \setminus \overline{\Gamma}_\alpha(0)$ and one of its dilates are pictured.

We will need the following maximum principle for $B \setminus \overline{\Gamma}_\alpha(0)$.

11.14 Lemma: *Let $\Omega = B \setminus \overline{\Gamma}_\alpha(0)$. Suppose u is real valued and continuous on $\overline{\Omega} \setminus \{0\}$, u is bounded and harmonic on Ω, and $u \leq M$ on $\partial\Omega \setminus \{0\}$. Then $u \leq M$ on Ω.*

PROOF: If $n > 2$, apply the maximum principle (Corollary 1.10) to the function $u(x) - M - \varepsilon|x|^{2-n}$ on Ω and let $\varepsilon \to 0$. When $n = 2$, the same argument applies with $\log 1/|x|$ in place of $|x|^{2-n}$. ∎



$B \setminus \overline{\Gamma}_\alpha(0)$ and one of its dilates.

With Ω as in Lemma 11.14, note that $r\Omega = (rB) \cap \Omega$ for every $r \in (0,1)$. This fact will be crucial in proving the next result.

11.15 Lemma: *Let $\Omega = B \setminus \overline{\Gamma}_\alpha(0)$. On $\partial\Omega$, set $f(x) = |x|$, and on $\overline{\Omega}$, set $u = \mathcal{P}[f]$. Then $-u$ is a barrier for Ω at 0.*

PROOF: The function u is harmonic on Ω, with $0 \le u \le 1$ on $\overline{\Omega}$. The external ball condition holds at each point of $\partial\Omega \setminus \{0\}$, so u is continuous and positive on $\overline{\Omega} \setminus \{0\}$. The proof will be completed by showing that $\limsup_{x \to 0} u(x) = 0$.

Fix $r \in (0,1)$. By the maximum principle, there exists a constant $c < 1$ such that $u \le c$ on $\partial(r\Omega) \setminus \{0\}$. Now define

$$v(x) = u(x) - \max\{r, c\} u(x/r)$$

for $x \in r\overline{\Omega}$. It is easy to check that $v \le 0$ on $\partial(r\Omega) \setminus \{0\}$. Applying Lemma 11.14, we obtain $v \le 0$ on $r\Omega$. Thus

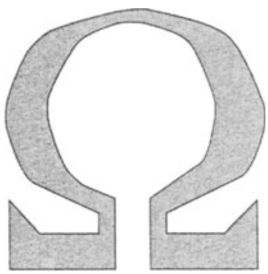
$$\limsup_{x\to 0} u(x) \le \max\{r,c\} \limsup_{x\to 0} u(x/r)$$
$$= \max\{r,c\} \limsup_{x\to 0} u(x).$$

Because $\max\{r,c\} < 1$, this implies that $\limsup_{x\to 0} u(x) = 0$, which completes the proof. ∎

The next result shows that the Dirichlet problem is solvable for every bounded domain satisfying the "external cone condition" at each of its boundary points. By a *cone* we shall mean any set of the form $a + T(\Gamma_\alpha^h(0))$, where $a \in \mathbf{R}^n$, T is a rotation, and $\Gamma_\alpha^h(0)$ is the truncation of $\Gamma_\alpha(0)$ defined in Chapter 7. We refer to a as the *vertex* of such a cone.

11.16 External Cone Condition: *If Ω is bounded and $\zeta \in \partial\Omega$ is the vertex of a cone contained in the complement of Ω, then Ω has a barrier at ζ.*

PROOF: Subharmonic functions are preserved by translations, dilations, and rotations, so without loss of generality we may assume that $\zeta = 0$ and that $\Omega \cap B \subset B \setminus \overline{\Gamma}_\alpha(0)$ for some $\alpha > 0$. Consider now the function $-u$ obtained in Lemma 11.15. This function is identically -1 on $S \setminus \overline{\Gamma}_\alpha(0)$. Thus if we extend this function by defining it to equal -1 on $\mathbf{R}^n \setminus B$, the result is a barrier for Ω at 0. ∎



An Ω satisfying the external cone condition.

Nonextendability Results

We now turn from the Dirichlet problem to another kind of boundary behavior question. Recall that if $\Omega \subset \mathbf{R}^2 = \mathbf{C}$, then there exists a function holomorphic on Ω that does not extend holomorphically to any larger set. This is usually proved by first showing that each discrete subset of Ω is the zero set of some function holomorphic on Ω. Because a discrete subset of Ω that clusters at each point of $\partial\Omega$ can always be chosen, some holomorphic function on Ω is nonextendable.

Remarkably, there exist domains in $\mathbf{C}^m$, $m > 1$, for which every holomorphic function extends across the boundary. For example, every holomorphic function on $\mathbf{C}^2 \setminus \{0\}$ extends to be entire on $\mathbf{C}^2$; see [12], pages 5-6, for details.

What about harmonic functions of more than two real variables? The next result shows that given any $\Omega \subset \mathbf{R}^n$, $n > 2$, a nonextendable positive harmonic function on Ω can always be produced.

11.17 Theorem: *Suppose $n > 2$ and $\Omega \subset \mathbf{R}^n$. Then there exists a positive harmonic function u on Ω such that*

$$\limsup_{x \to \zeta} u(x) = \infty$$

for every $\zeta \in \partial\Omega$.

PROOF: Assume first that Ω is connected. Let $\{\zeta_1, \zeta_2, \dots\}$ be a countable dense subset of $\partial\Omega$. Fixing $a \in \Omega$, we may choose positive constants c_m such that

$$c_m |a - \zeta_m|^{2-n} < 2^{-m}$$

for $m = 1, 2, \dots$. For $x \in \Omega$, define

$$u(x) = \sum_{m=1}^{\infty} c_m |x - \zeta_m|^{2-n}.$$

Each term in this series is positive and harmonic on Ω, and the series converges at $a \in \Omega$. By Harnack's Principle, u is harmonic on Ω, and we easily verify that u satisfies the conclusion of the theorem.

If Ω is not connected, we apply the preceding to each connected component of Ω to produce the desired function. ■

If $\Omega \subset \mathbf{R}^2$, it may not be possible to construct a positive harmonic function on Ω that is unbounded near every point of $\partial\Omega$. Recall, for example, that every positive harmonic function on $\mathbf{R}^2 \setminus \{0\}$ is constant (Corollary 3.3). Our next result, however, is valid for all $n \geq 2$.

11.18 Theorem: *Let $\Omega \subset \mathbf{R}^n$. Then there exists a real-valued harmonic function u on Ω such that*

$$\liminf_{x \to \zeta} u(x) = -\infty, \qquad \limsup_{x \to \zeta} u(x) = \infty$$

for every $\zeta \in \partial\Omega$.

PROOF: In the proof we will assume that $n > 2$; we leave the $n = 2$ case as an exercise.

Let I denote the set of isolated points of $\partial\Omega$. We assume that $\partial\Omega \setminus \bar{I}$ is nonempty; the proof that follows will easily adapt to the case $\partial\Omega = \bar{I}$. Select disjoint countable dense subsets D_- and D_+ of $\partial\Omega \setminus \bar{I}$, and write

$$D_- \cup D_+ \cup I = \{\zeta_1, \zeta_2, \dots\}.$$

Now choose pairwise disjoint compact sets $E_1, E_2, \dots$ such that for each m,

(a) $E_m \subset \Omega \cup \{\zeta_m\}$;

(b) ζ_m is a limit point of E_m.

For $\zeta_m \in I$ we insist that E_m be a closed ball of positive radius centered at ζ_m. For $\zeta_m \in D_- \cup D_+$ we will not be as fussy; for example, we can take E_m to be the union of $\{\zeta_m\}$ with a sequence in Ω converging to ζ_m. The pairwise disjointness of $\{E_m\}$ is easy to arrange by induction.

Set $v(x) = |x|^{2-n}$, $w(x) = x_1|x|^{-n}$, and for $m = 1, 2, \dots$, define

$$u_m(x) = \begin{cases} -v(x - \zeta_m) & \text{if } \zeta_m \in D_- \\ v(x - \zeta_m) & \text{if } \zeta_m \in D_+ \\ w(x - \zeta_m) & \text{if } \zeta_m \in I. \end{cases}$$

Note that u_m is harmonic on $\mathbf{R}^n \setminus \{\zeta_m\}$.

Choose compact sets $K_1, K_2, \dots \subset \Omega$ such that

$$K_1 \subset \operatorname{int} K_2 \subset K_2 \subset \cdots$$

and $\Omega = \cup K_m$. We may then choose positive constants c_m such that

$$c_m |u_m| \le 2^{-m}$$

on $K_m \cup E_1 \cup \cdots \cup E_{m-1}$. (Let E_0 be the empty set.)

Finally, define

$$u = \sum_{m=1}^{\infty} c_m u_m.$$

Because this series converges uniformly on compact subsets of Ω, u is harmonic on Ω.

To check the boundary behavior of u, we first consider the case where $\zeta \in \partial\Omega \setminus \bar{I}$. Let $\varepsilon > 0$. Then $B(\zeta, \varepsilon)$ contains some $\zeta_m \in D_-$. On the corresponding E_m, the series $\sum_{j \ne m} c_j u_j$ converges uniformly to a function continuous on E_m. Because $c_m u_m(x) \to -\infty$ as $x \to \zeta_m$ within E_m, the infimum of u over $B(\zeta, \varepsilon) \cap \Omega$ is $-\infty$. Similarly, the supremum of u over $B(\zeta, \varepsilon) \cap \Omega$ is ∞, giving us the desired conclusion.

Now suppose that $\zeta \in \bar{I}$. If $\varepsilon > 0$, then $B(\zeta, \varepsilon)$ contains some $\zeta_m \in I$. The series $\sum_{j \ne m} c_j u_j$ then converges uniformly to a continuous function on the closed ball E_m. Because $c_m u_m$ in this case maps any punctured ball $B(\zeta_m, r) \setminus \{\zeta_m\}$ onto $\mathbf{R}$, we are done. ∎

Note that if Ω is locally connected near $\partial\Omega$ (for example, if Ω is convex or has C^1-boundary), then the function u of Theorem 11.18 satisfies $u(B(\zeta, \varepsilon) \cap \Omega) = \mathbf{R}$ for every $\zeta \in \partial\Omega$ and every $\varepsilon > 0$. (Also see Exercise 22 of this chapter.)

Exercises

1. Suppose that Ω is a simply connected domain in the plane whose boundary is a Jordan curve. Explain how to use a suitable version of the Riemann mapping theorem to show that the Dirichlet problem for Ω is solvable.

2. Show that every translation, dilation, and rotation of a subharmonic function is subharmonic.

3. Suppose that u is subharmonic on Ω and that φ is increasing and convex on an open interval containing $u(\Omega)$. Prove that $\varphi \circ u$ is subharmonic on Ω.

4. Let u be a real-valued continuous function on Ω. Show that u is subharmonic on Ω if and only if for every compact $K \subset \Omega$ the following holds: if $u \le v$ on ∂K, where v is continuous on K and harmonic on the interior of K, then $u \le v$ on K.

5. Suppose that u is subharmonic on Ω and $a \in \Omega$. Show that the function $r \mapsto \int_S u(a + r\zeta)\, d\sigma(\zeta)$ is increasing. Conclude that the submean-value inequality 11.1 is valid for all $r < d(a, \partial\Omega)$.

6. Show that if a sequence of functions subharmonic on Ω converges uniformly on compact subsets of Ω, then the limit function is subharmonic on Ω.

7. Show that $|u|^p$ is subharmonic on Ω whenever u is harmonic on Ω and $1 \le p < \infty$. (This and Exercise 5 imply that $\|u_r\|_p$ is an increasing function of r, giving an alternative proof of Corollary 6.6.)

8. Suppose $u \in C^2(\Omega)$ is real valued. Use Taylor's Theorem to show that u is subharmonic on Ω if and only if $\Delta u \ge 0$ on Ω. (*Hint:* To show $\Delta u \ge 0$ implies u is subharmonic, first assume $\Delta u > 0$. Then consider the functions $u(x) + \varepsilon |x|^2$.)

9. Show that $|x|^p$ is subharmonic on $\mathbf{R}^n \setminus \{0\}$ for every $p > 2 - n$. Also show that $|x|^p$ is subharmonic on $\mathbf{R}^n$ for every $p > 0$.

10. (a) Show that a subharmonic function on $\mathbf{R}^2$ that is bounded above must be constant.

    (b) Suppose $n > 2$. Find a nonconstant subharmonic function on $\mathbf{R}^n$ that is bounded above.

11. Assume that f is holomorphic on $\Omega \subset \mathbf{R}^2 = \mathbf{C}$ and that u is subharmonic on $f(\Omega)$. Prove that $u \circ f$ is subharmonic on Ω.

12. Give an easier proof of Theorem 11.16 for the case $n = 2$.

13. With $n = 3$, let Ω denote the open unit ball with the x_3-axis removed. Show that the Dirichlet problem for Ω is not solvable.

14. Show that when $n = 2$, the Dirichlet problem is solvable for bounded open sets satisfying an "external segment condition". (*Hint:* An appropriate conformal map of $B_2 \setminus \{0\}$ onto the plane minus a line segment may be useful.)

15. Show that the Dirichlet problem is solvable for $B_3 \setminus (\overline{H}_2 \times \{0\})$.

16. (a) Give an example of a bounded Ω with C^1-boundary such that the external ball condition fails for some $\zeta \in \partial\Omega$.

    (b) Show that a domain with C^1-boundary satisfies the external cone condition at each of its boundary points.

17. Suppose Ω is bounded. Show that the Dirichlet problem for Ω is solvable if and only if $\mathcal{P}[-f] = -\mathcal{P}[f]$ on $\overline{\Omega}$ for every real-valued continuous f on $\partial\Omega$.

18. Suppose Ω is bounded and $a \in \Omega$. Show that there exists a unique positive Borel measure μ_a on $\partial\Omega$, with $\mu_a(\partial\Omega) = 1$, such that

$$\mathcal{P}[f](a) = \int_{\partial\Omega} f \, d\mu_a$$

for every real-valued continuous f on $\partial\Omega$. (The measure μ_a is called *harmonic measure* for Ω at a.)

19. Prove Theorem 11.18 in the case $n = 2$.

20. Show that if $\Omega \subset \mathbf{R}^2$ is bounded, then there exists a positive harmonic function u on Ω such that $\limsup_{x \to \zeta} u(x) = \infty$ for every $\zeta \in \partial\Omega$.

21. Suppose $\Omega \subset \mathbf{C}$ and $\{\zeta_1, \zeta_2, \ldots\}$ is a countable dense subset of $\partial\Omega$. Construct a holomorphic function on Ω of the form

$$\sum_{m=1}^{\infty} \frac{c_m}{z - \zeta_m}$$

that does not extend across any part of $\partial\Omega$. (This should be easier than the proof of Theorem 11.18.)

22. Given an arbitrary $\Omega \subset \mathbf{R}^n$, does there always exist a real-valued harmonic u on Ω such that $u(B(\zeta, \varepsilon) \cap \Omega) = \mathbf{R}$ for every $\zeta \in \partial\Omega$ and every $\varepsilon > 0$?

APPENDIX A

Volume, Surface Area, and Integration on Spheres

Volume of the Ball and Surface Area of the Sphere

In this section we compute the volume of the unit ball and surface area of the unit sphere in $\mathbf{R}^n$. Recall that $B = B_n$ denotes the unit ball in $\mathbf{R}^n$ and that $V = V_n$ denotes volume measure in $\mathbf{R}^n$. We begin by evaluating the constant $V(B)$, which appears in several formulas throughout the book.

A.1 Proposition: *The volume of the unit ball in* $\mathbf{R}^n$ *equals*

$$\frac{\pi^{n/2}}{(n/2)!} \qquad \textit{if } n \textit{ is even,}$$

$$\frac{2^{(n+1)/2}\pi^{(n-1)/2}}{1 \cdot 3 \cdot 5 \cdots n} \qquad \textit{if } n \textit{ is odd.}$$

PROOF: Assume $n > 2$, denote a typical point in $\mathbf{R}^n$ by (x, y), where $x \in \mathbf{R}^2$ and $y \in \mathbf{R}^{n-2}$, and express the volume $V_n(B_n)$ as an iterated integral:

$$\begin{aligned} V_n(B_n) &= \int_{B_n} 1\, dV_n \\ &= \int_{B_2} \int_{(1-|x|^2)^{1/2} B_{n-2}} 1\, dV_{n-2}(y)\, dV_2(x). \end{aligned}$$

The inner integral on the last line equals the $(n-2)$-dimensional volume of a ball in $\mathbf{R}^{n-2}$ with radius $(1-|x|^2)^{1/2}$. Thus

$$V_n(B_n) = V_{n-2}(B_{n-2}) \int_{B_2} (1-|x|^2)^{(n-2)/2}\, dV_2(x).$$

Switching to the usual polar coordinates in $\mathbf{R}^2$, we get

$$\begin{aligned} V_n(B_n) &= V_{n-2}(B_{n-2}) \int_{-\pi}^{\pi} \int_0^1 (1-r^2)^{(n-2)/2} r\, dr\, d\theta \\ &= \frac{2\pi}{n} V_{n-2}(B_{n-2}). \end{aligned}$$

The last formula can be easily used to prove the desired formula for $V_n(B_n)$ by induction in steps of 2, starting with the well-known results $V_2(B_2) = \pi$ and $V_3(B_3) = 4\pi/3$. ■

Readers familiar with the gamma function should be able to rewrite the formula given by A.1 as a single expression that holds whether n is even or odd (see Exercise 7 of this appendix).

Turning now to surface-area measure, we let S_n denote the unit sphere in $\mathbf{R}^n$.* Unnormalized surface-area measure on S_n will be denoted by s_n and normalized surface-area measure on S_n will be denoted by σ_n. Some of the arguments we give in the remainder of this appendix will be more intuitive than rigorous; the reader should have no trouble filling in the missing details. We presume some familiarity with surface-area measure.

Let us now find the relationship between $V_n(B_n)$ and $s_n(S_n)$. We do this with an old trick from calculus. For $h \approx 0$ we have

$$\begin{aligned} ((1+h)^n - 1)V_n(B_n) &= V_n((1+h)B_n) - V_n(B_n) \\ &\approx s_n(S_n)h. \end{aligned}$$

Dividing by h and letting $h \to 0$, we obtain $nV_n(B_n) = s_n(S_n)$. We record this result in the following proposition.

A.2 Proposition: *The unnormalized surface area of the unit sphere in $\mathbf{R}^n$ equals $nV_n(B_n)$.*

*A more common notation is S^{n-1}, which emphasizes that the sphere has dimension $n-1$ as a manifold. We use S_n to emphasize that the sphere lives in $\mathbf{R}^n$.

Slice Integration on Spheres

The map $\psi: B_{n-1} \to S_n$ defined by

$$\psi(x) = (x, \sqrt{1 - |x|^2})$$

parameterizes the upper hemisphere of S_n. The corresponding change of variables is given by the formula

A.3
$$ds_n(\psi(x)) = \frac{dV_{n-1}(x)}{\sqrt{1 - |x|^2}}.$$

Equation A.3 is found in most calculus texts in the cases $n = 2, 3$.

Consider now the map $\Psi: B_{n-k} \times S_k \to S_n$ defined by

$$\Psi(x, \zeta) = (x, \sqrt{1 - |x|^2}\,\zeta).$$

Here $1 \le k < n$. The map Ψ is one-to-one, and the range of Ψ is S_n minus a set that has s_n-measure 0 (namely, the set of points on S_n whose last k coordinates vanish). We wish to find the change of variables formula associated with Ψ.

Observe that $B_{n-k} \times S_k$ is an $(n-1)$-dimensional submanifold of $\mathbf{R}^n$ whose element of surface area is $d(V_{n-k} \times s_k)$. For fixed x, Ψ changes $(k-1)$-dimensional area on $\{x\} \times S_k$ by the factor $(1 - |x|^2)^{(k-1)/2}$. For fixed ζ, A.3 shows that Ψ changes $(n-k)$-dimensional area on $B_{n-k} \times \{\zeta\}$ by the factor $(1 - |x|^2)^{-1/2}$. Furthermore, the submanifolds $\Psi(\{x\} \times S_k)$ and $\Psi(B_{n-k} \times \{\zeta\})$ are perpendicular at their point of intersection, as is easily checked. The last statement implies that Ψ changes $(n-1)$-dimensional measure on $B_{n-k} \times S_k$ by the product of the factors above. In other words,

$$ds_n(\Psi(x, \zeta)) = (1 - |x|^2)^{(k-2)/2}\, dV_{n-k}(x)\, ds_k(\zeta).$$

The last equation and A.2 lead to the useful formula given in the next theorem. This formula shows how the integral over a sphere can be calculated by iterating an integral over lower-dimensional spherical slices. We state the formula in terms of normalized surface-area measure because that is what we have used most often.

A.4 Theorem: *Let f be a Borel measurable, integrable function on S_n. If $1 \le k < n$, then*

$$\int_{S_n} f\, d\sigma_n =$$

$$\frac{k}{n}\frac{V(B_k)}{V(B_n)} \int_{B_{n-k}} (1-|x|^2)^{\frac{k-2}{2}} \int_{S_k} f(x, \sqrt{1-|x|^2}\,\zeta)\, d\sigma_k(\zeta)\, dV_{n-k}(x).$$

Some cases of Theorem A.4 deserve special mention. We begin by choosing $k = n-1$, which is the largest permissible value of k. This corresponds to decomposing S_n into spheres of one less dimension by intersecting S_n with the family of hyperplanes orthogonal to the first coordinate axis. The ball B_{n-k} is just the unit interval $(-1,1)$, and so for $x \in B_{n-k}$ we can write x^2 instead of $|x|^2$. Thus we obtain the following corollary of Theorem A.4.

A.5 Corollary: *Let f be a Borel measurable, integrable function on S_n. Then*

$$\int_{S_n} f\, d\sigma_n =$$

$$\frac{n-1}{n}\frac{V(B_{n-1})}{V(B_n)} \int_{-1}^{1} (1-x^2)^{\frac{n-3}{2}} \int_{S_{n-1}} f(x, \sqrt{1-x^2}\,\zeta)\, d\sigma_{n-1}(\zeta)\, dx.$$

At the other extreme we can choose $k = 1$. This corresponds to decomposing S_n into pairs of points by intersecting S_n with the family of lines parallel to the nth coordinate axis. The sphere S_1 is the two-point set $\{-1, 1\}$, and $d\sigma_1$ is counting measure on this set, normalized so that each point has measure $1/2$. Thus we obtain the following corollary of Theorem A.4.

A.6 Corollary: *Let f be a Borel measurable, integrable function on S_n. Then*

$$\int_{S_n} f\, d\sigma_n =$$

$$\frac{1}{nV(B_n)} \int_{B_{n-1}} \frac{f(x, \sqrt{1-|x|^2}) + f(x, -\sqrt{1-|x|^2})}{\sqrt{1-|x|^2}}\, dV_{n-1}(x).$$

Let us now try $k = 2$ (assuming $n > 2$). Thus in A.4 the term $(1-|x|^2)^{(k-2)/2}$ disappears. The variable ζ in the formula given by

Theorem A.4 now ranges over the unit circle in $\mathbf{R}^2$, so we can replace ζ by $(\cos\theta, \sin\theta)$, which makes $d\sigma_2(\zeta)$ equal to $d\theta/(2\pi)$. Thus we obtain the following corollary of A.4.

A.7 Corollary ($n > 2$): *Let f be a Borel measurable, integrable function on S_n. Then*

$$\int_{S_n} f\,d\sigma_n =$$

$$\frac{1}{nV(B_n)} \int_{B_{n-2}} \int_{-\pi}^{\pi} f(x, \sqrt{1-|x|^2}\cos\theta, \sqrt{1-|x|^2}\sin\theta)\,d\theta\,dV_{n-2}(x).$$

An important special case of the last result occurs when $n = 3$. In this case B_{n-2} is just the interval $(-1, 1)$, and we get the following corollary.

A.8 Corollary: *Let f be a Borel measurable, integrable function on S_3. Then*

$$\int_{S_3} f\,d\sigma_3 = \frac{1}{4\pi} \int_{-1}^{1} \int_{-\pi}^{\pi} f(x, \sqrt{1-x^2}\cos\theta, \sqrt{1-x^2}\sin\theta)\,d\theta\,dx.$$

Exercises

1. Prove that

$$\int_{\mathbf{R}^n} \frac{1}{(|x|+1)^p}\,dV(x) < \infty$$

if and only if $p > n$.

2. (a) Consider the region on the unit sphere in $\mathbf{R}^3$ lying between two parallel planes that intersect the sphere. Show that the area of this region depends only on the distance between the two planes. (This result was discovered by the ancient Greeks.)

   (b) Show that the result in part (a) does not hold in $\mathbf{R}^n$ if $n \neq 3$ and "planes" are replaced by "hyperplanes".

3. Let f be a Borel measurable, integrable function on the unit sphere S_4 in $\mathbf{R}^4$. Define a function Ψ mapping the rectangular box $[-1,1] \times [-1,1] \times [-\pi,\pi]$ to S_4 by setting $\Psi(x,y,\theta)$ equal to

$$(x, \sqrt{1-x^2}\,y, \sqrt{1-x^2}\sqrt{1-y^2}\cos\theta, \sqrt{1-x^2}\sqrt{1-y^2}\sin\theta).$$

Prove that

$$\int_{S_4} f\,d\sigma_4 = \frac{1}{2\pi^2}\int_{-1}^{1}\sqrt{1-x^2}\int_{-1}^{1}\int_{-\pi}^{\pi} f(\Psi(x,y,\theta))\,d\theta\,dy\,dx.$$

4. Without writing down anything or using a computer, evaluate

$$\int_{S_7} {\zeta_1}^2\,d\sigma(\zeta).$$

5. Let m be a positive integer. Use A.5 to give an explicit formula for

$$\int_{-1}^{1} (1-x^2)^{m/2}\,dx.$$

6. Suppose that m is a positive integer.

   (a) Prove that

$$\int_{S_3} \zeta_1{}^m \, d\sigma(\zeta) = \begin{cases} \frac{1}{m+1} & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

   (b) Find a formula for $\int_{S_4} \zeta_1{}^m \, d\sigma(\zeta)$.

7. For readers familiar with the gamma function Γ: prove that the volume of the unit ball in $\mathbf{R}^n$ equals

$$\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}.$$

APPENDIX B

Harmonic Function Theory and Mathematica

Using the computational environment provided by *Mathematica*,* the authors have written software to manipulate many of the expressions that arise in the study of harmonic functions. This software allows the user to make symbolic calculations that would take a prohibitive amount of time if done without a computer. For example, Poisson integrals of polynomials can be computed exactly.

Our software for symbolic manipulation of harmonic functions is available over the internet without charge. It is distributed as a *Mathematica* package that will work on any computer that runs *Mathematica*. This *Mathematica* package and the instructions for using it are available at `http://math.sfsu.edu/axler/HFT_Math.html` and in the standard electronic mathematical archives (search for the files `HFT.m` and `HarmonicFunctionTheory.nb`). Comments, suggestions, and bug reports should be sent to `axler@sfsu.edu`.

Here are some of the capabilities of our *Mathematica* package:

- symbolic calculus in $\mathbf{R}^n$, including integration on balls and spheres;
- solution of the Dirichlet problem for balls, annular regions, and exteriors of balls in $\mathbf{R}^n$ (exact solutions with polynomial data);
- solution of the Neumann problem for balls and exteriors of balls in $\mathbf{R}^n$ (exact solutions with polynomial data);

***Mathematica* is a registered trademark of Wolfram Research, Inc.

- computation of bases for spaces of spherical harmonics in $\mathbf{R}^n$;
- computation of the Bergman projection for balls in $\mathbf{R}^n$;
- manipulations with the Kelvin transform K and the modified Kelvin transform $\mathcal{K}$;
- computation of the extremal function given by the Harmonic Schwarz Lemma (6.24) for balls in $\mathbf{R}^n$;
- computation of harmonic conjugates in $\mathbf{R}^2$.

New features are frequently added to this software.

References

Harmonic function theory has a rich history and a continuing high level of research activity. *MathSciNet* lists over five thousand papers published since 1940 for which the review contains the phrase "harmonic function", with over fifteen hundred of these papers appearing since 1990. Although we have drawn freely from this heritage, we list here only those works cited in the text.

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Symbol Index

Symbols are sorted by ignoring everything except Latin and Greek letters. For sorting purposes, Greek letters are assumed to be spelled out in full with Latin letters. For example, Ω_E is translated to "OmegaE", which is then sorted with other entries beginning with "O" and before $O(n)$, which translates to "On". The symbols that contain no Latin or Greek letters appear first, sorted by page number.

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